Uniform and Composite Solids

Specification: AQA 7367: ME2 (uniform bodies, composite bodies, standard results) · Edexcel 9FM0: FM2 Further centres of mass (solids)

INDEPENDENT

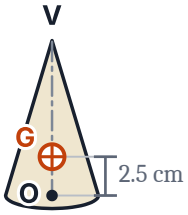
You do

A · Standard results Formula booklet: cone h/4 from the base, hemisphere 3r/8, hemispherical shell r/2, conical shell h/3.

1 A uniform solid cone has height 10 cm. Find the distance of its centre of mass from
(a) its base
(b) its vertex.

(a) $h/4 = 2.5$ cm — cone: h/4 from the base

(b) $10 - 2.5 = 7.5$ cm — from the vertex: 3h/4



picture: G

2 A uniform solid hemisphere of radius 6 cm rests with its plane face on a table. Find the height of its centre of mass above the table.

$3r/8 = 18/8 = 2.25$ cm above the plane face, which is on the table — 3r/8 from the plane face



picture: G

3 A uniform hemispherical bowl (a thin shell) has radius 9 cm. Find the distance of its centre of mass from the plane of its rim.

$r/2 = 4.5$ cm — shell: r/2

4 A uniform conical shell, open at its base, has height 9 cm. It stands on its rim on a table. Find the height of its centre of mass above the table.

$h/3 = 3$ cm — conical shell: h/3 from the rim

B · Composite solids Table: V, x, Vx. A part removed has negative volume.

5 A toy is a uniform solid hemisphere of radius 2 cm with a uniform solid cone of base radius 2 cm and height 6 cm on its plane face (centre O). Find the distance of the centre of mass from O and from the vertex.

	V	x	Vx
cone	8π	$3/2$	12π
hemisphere	$16\pi/3$	$-3/4$	-4π
Σ	$\mathbf{40\pi/3}$		$\mathbf{8\pi}$

table: V, x, Vx

$\bar{x} = 8\pi \div (40\pi/3) = 3/5 = 0.6$ cm above O — ΣVx ÷ ΣV

From the vertex: $6 - 0.6 = 5.4$ cm — distance from the vertex



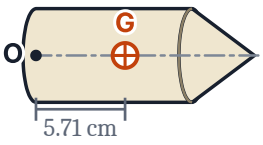
picture: G just above O

6 A uniform solid is a cylinder of radius 3 cm and length 10 cm with a cone of base radius 3 cm and height 4 cm on one end. Find the distance of its centre of mass from the plane end of the cylinder.

	V	x	Vx
cylinder	90π	5	450π
cone	12π	11	132π
Σ	$\mathbf{102\pi}$		$\mathbf{582\pi}$

cone: 10 + 4/4 = 11

$\bar{x} = 582\pi \div 102\pi = 97/17 = 5.71$ cm — ΣVx ÷ ΣV



picture: G

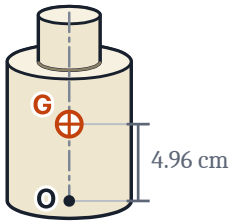
7 A uniform solid cylinder of radius 2 cm and height 3 cm stands centrally on top of a uniform solid cylinder of radius 4 cm and height 9 cm. Find the height of the centre of mass above the base.

	V	x	Vx
lower	144π	$9/2$	648π
upper	12π	$21/2$	126π
Σ	$\mathbf{156\pi}$		$\mathbf{774\pi}$

centres: 4.5 and 9 + 1.5

$\bar{x} = 774\pi \div 156\pi = 129/26 = 4.96$ cm

ΣVx ÷ ΣV



picture: G

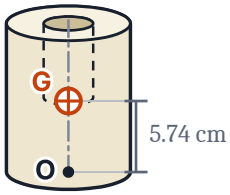
8 A uniform solid cylinder has radius 5 cm and height 12 cm. A cylindrical hole of radius 2 cm and depth 6 cm is drilled into its top face along its axis. Find the height of the centre of mass of the remaining solid above the base.

	V	x	Vx
cylinder	300π	6	1800π
hole	-24π	9	-216π
Σ	$\mathbf{276\pi}$		$\mathbf{1584\pi}$

hole: negative volume

$\bar{x} = 1584\pi \div 276\pi = 132/23 = 5.74$ cm

ΣVx ÷ ΣV



picture: G below the middle

C · Reasoning No calculator. Explain in words.

9 A uniform solid cone and a uniform conical shell have the same height. Which has its centre of mass nearer to its base? Explain why.

Solid: $h/4$ from the base; shell: $h/3$. The solid cone's G is nearer the base.

compare h/4 and h/3

In the solid, most of the volume is near the base (the cross-sections are largest there); in the shell, the area is spread more evenly up the slant.

where the mass is

10 A cylindrical hole is drilled along the whole length of the axis of a uniform solid cylinder. Where is the centre of mass of the remaining solid? Explain.

The remaining solid is still symmetric about its axis and about the plane halfway along it, so G is on both: at the centre of the cylinder.

symmetry: axis and mid-plane

11 A toy is a hemisphere with a cone of the same radius on top (as in Worked example 5b). Tom says: "Making the cone taller makes the toy more stable." Is he right? For a radius of 6 cm, find the greatest height of the cone for which the toy returns upright.

No. A taller cone has more volume, higher up: G rises. Once G is above O the toy falls over.

G rises with h

Upright when $h < r\sqrt{3}$ (Worked example 5b):
$h < 6\sqrt{3} = 10.4$ cm

h < r√3

D · Show that, unknowns, densities and shells Algebra in the table. Masses for different materials; areas for shells.

12 A frustum is made by cutting a uniform solid cone of base radius $2r$ and height $2h$ parallel to its base, halfway up, and removing the top cone (radius r, height h). Show that the centre of mass of the frustum is at a distance $\dfrac{11h}{28}$ from its larger plane face.

Large cone $\frac{1}{3}\pi(2r)^2(2h) = 8\pi r^2h/3$ at $2h/4 = h/2$; top cone $\pi r^2h/3$ at $h + h/4 = 5h/4$

cone minus cone

$\Sigma Vx = (8\pi r^2h/3)(h/2) - (\pi r^2h/3)(5h/4) = 11\pi r^2h^2/12$

moments about the base

$\Sigma V = 7\pi r^2h/3$, $\bar{x} = (11h/12) \div (7/3) = 11h/28$

÷ ΣV

13 A toy is a uniform solid hemisphere of radius 6 cm with a uniform solid cone of radius 6 cm and height h cm on its plane face (centre O). The centre of mass of the toy is 1 cm below O. Using Worked example 5b, find h.

$\dfrac{h^2 - 3 \times 36}{4(h + 12)} = -1$

$\bar{x} = -1$

$h^2 - 108 = -4h - 48 \Rightarrow h^2 + 4h - 60 = 0$

rearrange

$(h + 10)(h - 6) = 0 \Rightarrow h = 6$

h > 0

14 A toy is made of a uniform solid hemisphere of radius 2 cm and a uniform solid cylinder of radius 2 cm and height 4 cm joined at their plane faces (centre O). The hemisphere is made of material three times as dense as the cylinder.
(a) Find the distance of the centre of mass from O.
(b) The toy rests with its curved surface on a horizontal floor. Does it return upright when tilted? Explain.

(a) Masses ∝ density × volume: cylinder $16\pi\rho$ at 2; hemisphere $3\rho \times 16\pi/3 = 16\pi\rho$ at $-3/4$

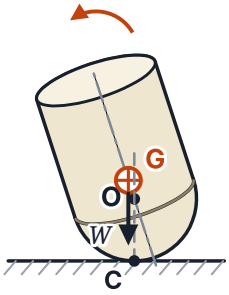
m = ρV

$\bar{x} = (32\pi\rho - 12\pi\rho) \div 32\pi\rho = 5/8 = 0.625$ cm above O

Σmx ÷ Σm

(b) No: G is above O, so the weight turns it further over.

G above O: falls



picture: G above O, falls

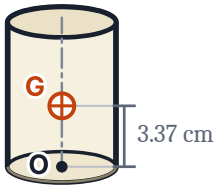
15 An open-topped can is made of thin uniform sheet metal: the curved surface of a cylinder of radius 3 cm and height 8 cm, and a circular base. Find the height of its centre of mass above the base.

Curved surface $2\pi rh = 48\pi$ at 4; base 9π at 0

mass ∝ area

$\bar{x} = 192\pi \div 57\pi = 64/19 = 3.37$ cm

ΣAx ÷ ΣA



picture: G

E · Exam-style question Show your method: marks are for working.

16 A toy is a uniform solid hemisphere of radius 3 cm with a uniform solid cone of base radius 3 cm and height h cm on its plane face, centre O.

(a) Taking x upwards from O, show that $\bar{x} = \dfrac{h^2 - 27}{4(h+6)}$ cm (negative: below O). [4 marks]

(b) The toy rests on its curved surface on a horizontal floor. Find the values of h for which it returns upright when tilted. [2 marks]

(c) Given $h = 2$, it hangs freely from a point A on the rim of the plane face. Find the angle between its axis and the vertical. [2 marks]

(a) Cone $3\pi h$ at $h/4$; hemisphere 18π at $-9/8$ **B1**

volumes and centres

$\Sigma Vx = 3\pi(h^2 - 27)/4$, $\Sigma V = 3\pi(h+6)$ **M1 M1**

ΣVx and ΣV

$\bar{x} = \dfrac{h^2 - 27}{4(h+6)}$ **A1**

÷ ΣV

(b) G below O: $h^2 - 27 < 0$, so $0 < h < 3\sqrt{3}$ (5.20) **M1 A1**

G below O

(c) $\bar{x} = -23/32$: $\tan\theta = 3 \div (23/32) = 96/23$, $\theta = 76.5°$ **M1 A1**

tan θ = r ÷ OG