Unit Review

Specification: AQA 7367: ME1, ME2 · Edexcel 9FM0: FM2 Centres of mass of plane figures, Further centres of mass

INDEPENDENT Mixed practice

A · Fluency One or two steps each. Which tool?

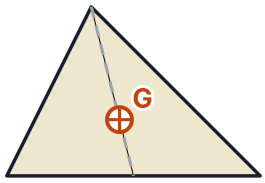
1 Particles of mass 4 kg, 1 kg and 5 kg lie on the x-axis at $x = -2$, $x = 3$ and $x = 4$. Find $\bar{x}$.

$\Sigma m = 10$, $\Sigma mx = -8 + 3 + 20 = 15$ from L1 — Σmx with signs

$\bar{x} = 15 \div 10 = 1.5$ — Σmx ÷ Σm

2 A uniform triangular lamina has vertices $(0, 0)$, $(9, 0)$ and $(3, 6)$. Find its centre of mass.

Triangle: G is the mean of the three vertices from L3 — triangle: mean of the vertices



picture: G on the median

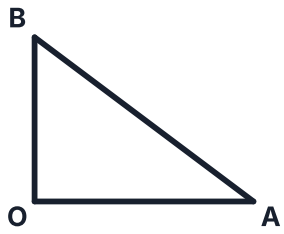
$((0 + 9 + 3) \div 3, (0 + 0 + 6) \div 3) = (4, 2)$ — add and ÷ 3

3 (a) A uniform solid cone has height 20 cm.
(b) A uniform conical shell, open at its base, has height 18 cm.
For each, find the distance of the centre of mass from the vertex.

(a) $h/4 = 5$ from the base, so $20 - 5 = 15$ cm from the vertex from L5 — solid: h/4 from the base

(b) $h/3 = 6$ from the rim, so $18 - 6 = 12$ cm from the vertex — shell: h/3 from the rim

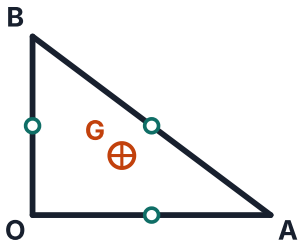
4 A uniform wire is bent into a triangle OAB with $O(0, 0)$, $A(4, 0)$ and $B(0, 3)$. Find the centre of mass of the wire framework.



$OA = 4$, $OB = 3$, $AB = 5$; each mass at its midpoint from L2 — lengths (3, 4, 5)

	L	x	y	Lx	Ly
OA	4	2	0	8	0
OB	3	0	1.5	0	4.5
AB	5	2	1.5	10	7.5
Σ	**12**			**18**	**12**

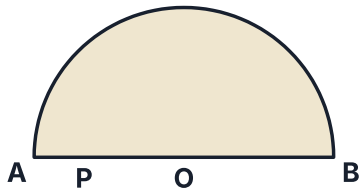
table: L, x, y, Lx, Ly



picture: G inside the frame

$\bar{x} = 18 \div 12 = 1.5$, $\bar{y} = 12 \div 12 = 1$: G is at $(1.5, 1)$ — ΣLx ÷ ΣL

5 A uniform semicircular lamina has diameter AB = 12 cm and centre O. It is freely suspended from the point P on AB with AP = 2 cm. Find the angle between AB and the vertical.



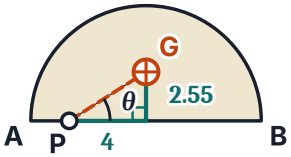
G is on the axis of symmetry, $4r/3\pi = 8/\pi$ cm from O

from L3, L4

semicircle: 4r ÷ 3π

From P: along AB $PO = 4$; across $8/\pi = 2.55$

along and across from P



picture: triangle from P

$\tan\theta = (8/\pi) \div 4 = 2/\pi$, $\theta = 32.5°$

tan θ = across ÷ along

6 Particles of mass 3 kg and m kg are on the x-axis at $x = 1$ and $x = 6$. Their centre of mass is at $x = 4$. Find m.

$(3 + m) \times 4 = 3 \times 1 + 6m$ from L1

(Σm) x̄ = Σmx

$12 + 4m = 3 + 6m \Rightarrow m = 4.5$

solve

B · Multi-step Several tools in one question. Sketch first.

7 A uniform lamina is made of the rectangle OABC, with $O(0, 0)$, $A(8, 0)$, $B(8, 4)$ and $C(0, 4)$, and a semicircle with diameter AB on the outside of the rectangle.
(a) Find the centre of mass of the lamina.
(b) The lamina is freely suspended from O. Find the angle between OA and the vertical.



(a) Rectangle 32 at $x = 4$; semicircle 2π at $x = 8 + 8/(3\pi)$

from L3, L4

semicircle: 4r ÷ 3π beyond AB

	A	x	Ax
rect.	32	4	128
semi.	2π	$8 + 8/(3\pi)$	$16\pi + 16/3$
Σ	$\mathbf{32 + 2\pi}$		$\mathbf{400/3 + 16\pi}$

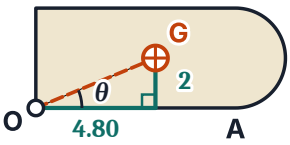
table: A, x, Ax

$\bar{x} = (400/3 + 16\pi) \div (32 + 2\pi) = 4.80$; $\bar{y} = 2$ by symmetry

ΣAx ÷ ΣA; symmetry

(b) G below O: along OA 4.796, across 2

along and across from O



picture: triangle from O

$\tan\theta = 2 \div 4.796$, $\theta = 22.6°$

tan θ = across ÷ along

8 A uniform wire of length 16 cm is bent into three sides AB, BC and CD of a rectangle, with AB = CD = 4 cm and BC = 8 cm.
(a) Find the distance of the centre of mass from BC.
(b) It hangs freely from A. Find the angle between AB and the vertical.



(a) By symmetry G is on the perpendicular bisector of BC. Distances from BC: AB and CD at 2, BC at 0 from L2, L4

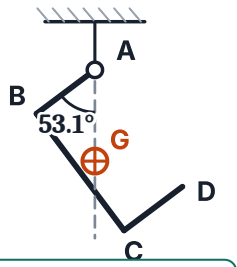
symmetry; midpoints

$\bar{y} = (4 \times 2 + 8 \times 0 + 4 \times 2) \div 16 = 1$ cm from BC

ΣLy ÷ ΣL

(b) From A, along AB: $4 - 1 = 3$; across (half of BC): 4

along and across from A

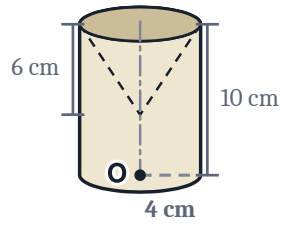


picture: G below A

$\tan\theta = 4/3$, $\theta = 53.1°$

tan θ = across ÷ along

9 A cone of base radius 4 cm and height 6 cm is drilled out of a uniform solid cylinder of radius 4 cm and height 10 cm. The base of the cone is the top face of the cylinder and they have the same axis.
(a) Find the distance of the centre of mass of the remaining solid from the bottom face.
(b) The solid is freely suspended from a point on the rim of its bottom face. Find the angle between its axis and the vertical.



(a) x up from O, the centre of the bottom face. Cylinder 160π at 5; cone $\frac{1}{3}\pi \times 16 \times 6 = 32\pi$ removed, at $10 - 6/4 = 8.5$ from L4, L5

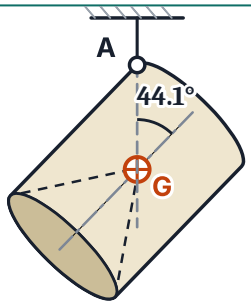
removed: negative volume

	V	x	Vx
cylinder	160π	5	800π
cone	-32π	$17/2$	-272π
Σ	$\mathbf{128\pi}$		$\mathbf{528\pi}$

table: V, x, Vx

$\bar{x} = 528\pi \div 128\pi = 33/8 = 4.125$ cm

ΣVx ÷ ΣV



picture: G below the point

(b) $\tan\theta = 4 \div (33/8) = 32/33$, $\theta = 44.1°$

tan θ = r ÷ OG

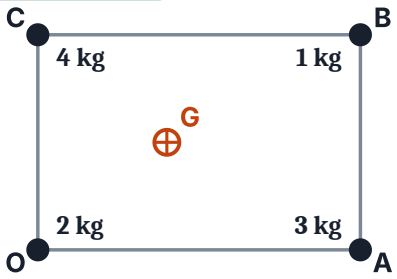
10 A light rectangular lamina OABC has $O(0,0)$, $A(6,0)$, $B(6,4)$ and $C(0,4)$. Particles of mass 2 kg, 3 kg, 1 kg and 4 kg are attached at O, A, B and C.
(a) Find the centre of mass of the system.
(b) An extra particle of mass m kg is attached at B, so that the centre of mass lies on the line $x = 3$. Find m.

(a) Light lamina: particles only. $\Sigma m = 10$, $\Sigma mx = 18 + 6 = 24$, $\Sigma my = 4 + 16 = 20$ from L1

light: particles only

G is at $(2.4, 2)$

Σmx ÷ Σm, Σmy ÷ Σm



picture: G at (2.4, 2)

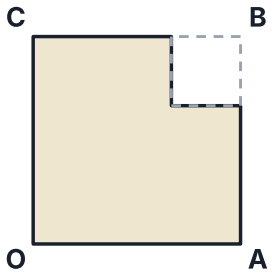
(b) $(10 + m) \times 3 = 24 + 6m$

(Σm) x̄ = Σmx

$30 + 3m = 24 + 6m \Rightarrow m = 2$

solve

11 A uniform square lamina OABC has $O(0,0)$, $A(6,0)$, $B(6,6)$ and $C(0,6)$. The square $4 \le x \le 6$, $4 \le y \le 6$ is cut away.
(a) Find the centre of mass of the remaining lamina.
(b) It is freely suspended from O. Without further calculation, find the angle between OA and the vertical.



(a) Square 36 at $(3,3)$; cut away -4 at $(5,5)$ from L3, L4

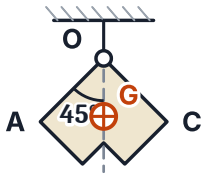
removed: negative area

$\bar{x} = (108 - 20) \div 32 = 2.75$; $\bar{y} = 2.75$ by symmetry in $y = x$

ΣAx ÷ ΣA; symmetry

G is at $(2.75, 2.75)$, on the diagonal OB

G on the diagonal



picture: OB vertical

(b) G is on OB, so OB hangs vertically below O.

G below O: OB vertical

OA makes 45° with OB, so 45° with the vertical.

square: 45°

C · Reasoning No calculator. Explain in words.

12 Ana says: "A uniform triangular wire framework and a uniform triangular lamina of the same shape always have the same centre of mass." Test her claim with the triangle $O(0,0)$, $A(6,0)$, $B(0,8)$.

Lamina: the mean of the vertices, $(2, 8/3)$ from L2, L3

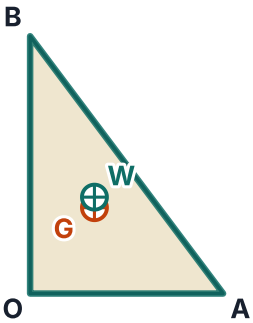
lamina: mean of the vertices

Framework: $OA = 6$ at $(3,0)$, $OB = 8$ at $(0,4)$, $AB = 10$ at $(3,4)$

wire: mass at midpoints

$\bar{x} = 48 \div 24 = 2$, $\bar{y} = 72 \div 24 = 3$

ΣLx ÷ ΣL



picture: lamina G and wire W

Different: $(2,3) \neq (2, 8/3)$. Ana is wrong (they agree only for an equilateral triangle).

counter-example

13 Each of these is uniform with radius 8 cm: a semicircular arc of wire, a semicircular lamina, a solid hemisphere and a hemispherical shell. Put them in order of the distance of their centre of mass from the centre O, largest first. Use $3 < \pi < 3.2$.

Arc $2r/\pi = 16/\pi$; lamina $4r/3\pi = 32/(3\pi)$; solid $3r/8 = 3$; shell $r/2 = 4$ from L2, L3, L5

four standard results

$\pi < 4$, so $16/\pi > 4$. $3\pi > 8$, so $32/(3\pi) < 4$; $9\pi < 32$, so $32/(3\pi) > 3$.

compare, using the bounds

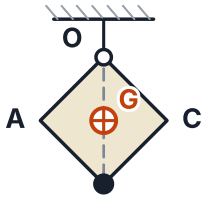
Arc, shell, lamina, solid hemisphere.

largest first

14 A uniform square lamina hangs freely from one corner, so that a diagonal is vertical. A particle is then attached to the lowest corner. Does the position in which the lamina hangs change? Explain.

The lamina's G is on the vertical diagonal, and so is the particle from L1, L4

both on the diagonal



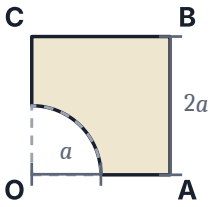
picture: G and the particle on the diagonal

The new centre of mass is between them, on the same diagonal, so still vertically below the corner: the lamina hangs exactly as before.

G stays on the line

D · Show that Algebra in the table. Leave π in.

15 A uniform square lamina OABC has side $2a$, with O at the origin, OA along the x-axis and OC along the y-axis. A quarter-circle of radius a and centre O is cut away. Show that the centre of mass of the remaining lamina is at a distance $\dfrac{44a}{3(16-\pi)}$ from OC.



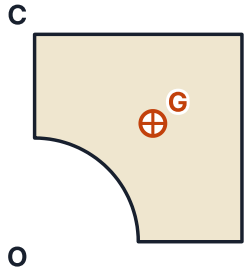
Quarter circle: a sector with $\alpha = \pi/4$. Its centre is $2a\sin(\pi/4) \div (3\pi/4) = 4\sqrt{2}a/(3\pi)$ from O along $y = x$, so $x = 4a/(3\pi)$ from L3

sector: 2r sin α ÷ 3α

Square $4a^2$ at $x = a$; quarter circle $-\pi a^2/4$ at $x = 4a/(3\pi)$

removed: negative area

$\Sigma Ax = 4a^3 - a^3/3 = 11a^3/3$, $\Sigma A = a^2(16-\pi)/4$ ΣAx and ΣA



picture: G on the diagonal

$$\bar{x} = \frac{11a^3/3}{a^2(16-\pi)/4} = \frac{44a}{3(16-\pi)}$$

÷ ΣA

16 A toy is made of a uniform solid hemisphere of radius r and density 2ρ, joined at its plane face (centre O) to a uniform solid cone of base radius r, height h and density ρ. Taking x upwards from O along the axis, show that $\bar{x} = \dfrac{h^2 - 6r^2}{4(h+4r)}$. Hence find the condition on h for the toy to return upright when it rests on its curved surface.

Masses $m = \rho V$: cone $\frac{1}{3}\pi r^2 h\rho$ at $h/4$; hemisphere $2\rho \times \frac{2}{3}\pi r^3 = 4\pi r^3\rho/3$ at $-3r/8$ from L1, L5

different densities: m = ρV

$\Sigma mx = \pi r^2\rho(h^2/12 - r^2/2) = \pi r^2\rho(h^2 - 6r^2)/12$

Σmx: factorise

$\Sigma m = \pi r^2\rho(h+4r)/3$

Σm: factorise

$$\bar{x} = \frac{h^2 - 6r^2}{4(h+4r)}$$

÷ Σm; πr²ρ cancels

Upright when G is below O: $h^2 - 6r^2 < 0$, so $h < r\sqrt{6}$

G below O

E · Exam-style questions Show your method: marks are for working.

17 *AQA-style*. Lengths are in cm. A uniform lamina is made from the rectangle OABC, where $O(0,0)$, $A(12,0)$, $B(12,6)$ and $C(0,6)$, by cutting away a semicircle of radius 2 cm whose diameter lies along OC, with its centre at $(0,3)$.

(a) Show that the centre of mass of the lamina is at a distance $\dfrac{640}{3(36-\pi)}$ cm from OC. [4 marks]

(b) The lamina is freely suspended from B. Find the angle between BA and the vertical. [2 marks]

(c) The lamina has mass M. A particle of mass kM is attached at C. When the loaded lamina is freely suspended from O, OA makes an angle of 45° with the vertical. Find k. [2 marks]

(a) Rectangle 72 at $x = 6$; semicircle -2π at $x = 4r/3\pi = 8/(3\pi)$; $\bar{y} = 3$ by symmetry from L1, L3, L4 **B1 B1** — areas and centres

$\Sigma Ax = 432 - 16/3 = 1280/3$, $\Sigma A = 72 - 2\pi$ **M1** — ΣAx and ΣA

$\bar{x} = \dfrac{1280}{3(72-2\pi)} = \dfrac{640}{3(36-\pi)} = 6.49$ **A1** — ÷ ΣA

(b) From B, along BA: $6 - 3 = 3$; across: $12 - 6.49 = 5.51$ **M1** — along and across from B

$\tan\theta = 5.51 \div 3$, $\theta = 61.4°$ **A1** — tan θ = across ÷ along

(c) OA at 45°: G is on $y = x$. New G: $x = 6.49/(1+k)$, $y = (3+6k)/(1+k)$ **M1** — G on y = x

$x = y$: $\dfrac{640}{3(36-\pi)} = 3 + 6k \Rightarrow k = 0.582$ **A1** — x = y; (1 + k) cancels

18 *Edexcel-style*. A uniform solid cylinder has radius 3 cm and height 8 cm. A cone of base radius 3 cm and height 4 cm is removed from one end: the base of the cone is that end face and they have the same axis. O is the centre of the other end face.

(a) Show that the centre of mass of the remaining solid is 3.4 cm from O. [4 marks]

(b) The solid is freely suspended from a point A on the rim of the face containing O. Find the angle between the axis and the vertical. [2 marks]

(c) The solid has mass M. A particle of mass kM is attached at O. When the loaded solid is freely suspended from A, its axis makes an angle of 45° with the vertical. Find k. [2 marks]

(a) Cylinder 72π at 4; cone $\frac{1}{3}\pi \times 9 \times 4 = 12\pi$ removed, at $8 - 4/4 = 7$ from L1, L4, L5 **B1 B1** — volumes and centres

	V	x	Vx
cylinder	72π	4	288π
cone	-12π	7	-84π
Σ	**60π**		**204π**

table: V, x, Vx

M1

$\bar{x} = 204\pi \div 60\pi = 3.4$ cm from O **A1** — ÷ ΣV

(b) $\tan\theta = 3 \div 3.4$, $\theta = 41.4°$ **M1 A1** — tan θ = r ÷ OG

(c) 45°: $\tan\theta = 1$, so the new $OG = 3$. $(M + kM) \times 3 = M \times 3.4 + kM \times 0$ **M1** — 45°: OG = r

$3 + 3k = 3.4 \Rightarrow k = 2/15$ **A1** — solve