Laminas by Integration

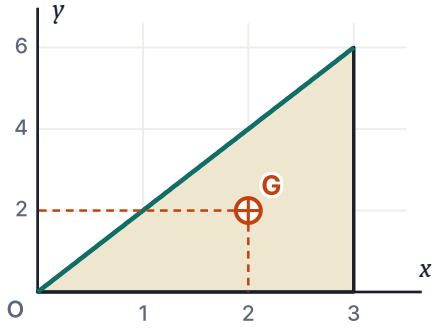
Specification: AQA 7367: ME3 (lamina by integration) · Edexcel 9FM0: FM2 Further centres of mass (laminas and non-uniform rods by integration)

INDEPENDENT You do

A · Under a curve $A = \int y\,dx$, then $\bar{x} = \int xy\,dx \div A$ and $\bar{y} = \int \tfrac{1}{2}y^2\,dx \div A$.

1 A uniform lamina is bounded by $y = 2x$, the x-axis and $x = 3$. Find its centre of mass by integration.

$A = \int_0^3 2x\,dx = 9$; $\int_0^3 2x^2\,dx = 18$: $\bar{x} = 2$ — area, then $\int xy\,dx$



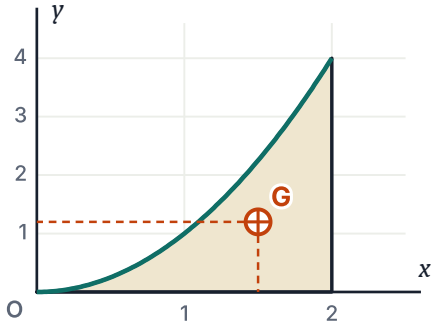
picture: G

$\int \tfrac{1}{2}y^2\,dx = \int_0^3 \tfrac{1}{2}(2x)^2\,dx = \int_0^3 2x^2\,dx = 18$, $\bar{y} = 18 \div 9 = 2$. G $(2, 2)$

$\int \tfrac{1}{2}y^2\,dx \div A$

2 A uniform lamina is bounded by $y = x^2$, the x-axis and $x = 2$. Find its centre of mass.

$A = 8/3$, $\int_0^2 x^3\,dx = 4$: $\bar{x} = 4 \div (8/3) = 3/2$ — area, then $\int xy\,dx$



picture: G

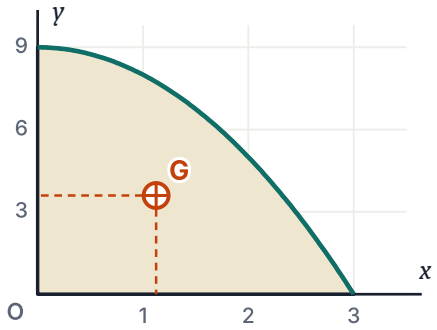
$\int_0^2 \tfrac{1}{2}x^4\,dx = 16/5$, $\bar{y} = (16/5) \div (8/3) = 6/5$. G $(1.5, 1.2)$

$\int \tfrac{1}{2}y^2\,dx \div A$

3 A uniform lamina is bounded by $y = 9 - x^2$ and the coordinate axes, with $x \geq 0$. Find its centre of mass.

$A = \int_0^3 (9 - x^2)\,dx = 27 - 9 = 18$ — area

$\int_0^3 (9x - x^3)\,dx = 81/2 - 81/4 = 81/4$, $\bar{x} = 9/8$ — $\int xy\,dx \div A$



picture: G

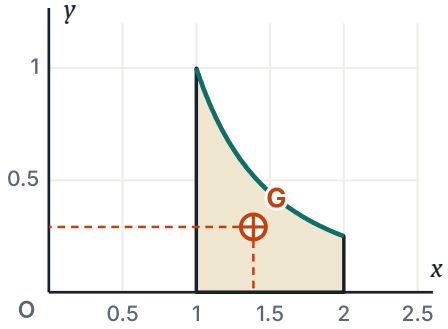
$\int_0^3 \tfrac{1}{2}(81 - 18x^2 + x^4)\,dx = \tfrac{1}{2}(243 - 162 + 243/5) = 324/5$, $\bar{y} = 18/5$. G $(1.125, 3.6)$

expand, then $\int \tfrac{1}{2}y^2\,dx$

4 A uniform lamina is bounded by $y = 1/x^2$, the x-axis and the lines $x = 1$ and $x = 2$. Find its centre of mass, giving $\bar{x}$ exactly.

$A = \int_1^2 x^{-2}\,dx = \left[-x^{-1}\right]_1^2 = \tfrac{1}{2}$ — area

$\int_1^2 x^{-1}\,dx = \ln 2$, $\bar{x} = 2\ln 2 = 1.39$ — $\int 1/x\,dx = \ln x$



picture: G

$\int_1^2 \tfrac{1}{2}x^{-4}\,dx = \left[-x^{-3}/6\right]_1^2 = 7/48$, $\bar{y} = (7/48) \div \tfrac{1}{2} = 7/24 = 0.292$

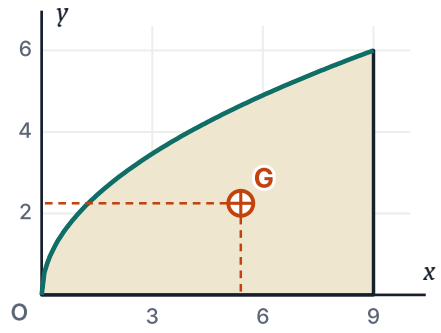
$\int \tfrac{1}{2}y^2\,dx \div A$

B · Roots, two curves, signs and rods Strip between two curves: height $y_1 - y_2$, centre $\frac{1}{2}(y_1 + y_2)$.

5 A uniform lamina is bounded by $y = 2\sqrt{x}$, the x-axis and $x = 9$. Find its centre of mass.

$A = \int_0^9 2x^{1/2}\,dx = (4/3)(27) = 36$ — write √x as a power

$\int_0^9 2x^{3/2}\,dx = (4/5)(243) = 972/5$, $\bar{x} = 27/5 = 5.4$ — ∫xy dx ÷ A



picture: G

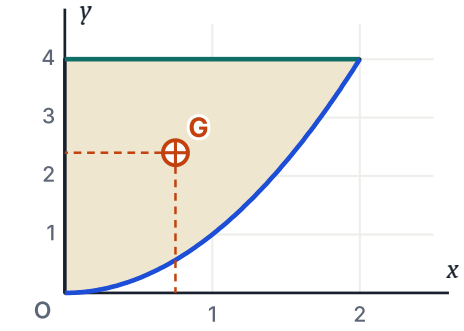
$\int_0^9 2x\,dx = 81$, $\bar{y} = 81/36 = 9/4$. G $(5.4, 2.25)$ — ½y² = 2x

6 A uniform lamina occupies the region between the curve $y = x^2$, the line $y = 4$ and the y-axis $(0 \le x \le 2)$. Find its centre of mass.

Strip height $4 - x^2$: $A = 16/3$, $\int_0^2 (4x - x^3)\,dx = 4$, $\bar{x} = 3/4$

top − bottom

$\int_0^2 \frac{1}{2}(16 - x^4)\,dx = \frac{1}{2}(32 - 32/5) = 64/5$ — ½(y₁² − y₂²)



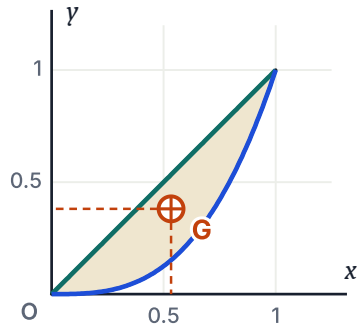
picture: G

$\bar{y} = (64/5) \div (16/3) = 12/5$. G $(0.75, 2.4)$ — ÷ A

7 A uniform lamina occupies the region between $y = x$ and $y = x^3$ for $0 \le x \le 1$. Find its centre of mass.

$A = \int_0^1 (x - x^3)\,dx = \frac{1}{2} - \frac{1}{4} = 1/4$ — top − bottom

$\int_0^1 (x^2 - x^4)\,dx = 2/15$, $\bar{x} = 8/15$ — ∫x(y₁ − y₂) dx ÷ A



picture: G

$\int_0^1 \frac{1}{2}(x^2 - x^6)\,dx = 2/21$, $\bar{y} = 8/21$. G $(0.533, 0.381)$

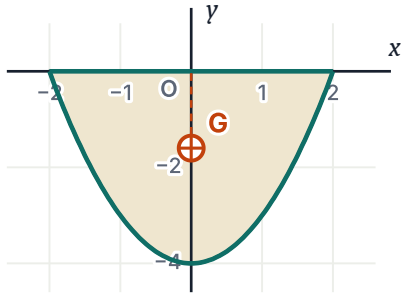
½(y₁² − y₂²)

8 A uniform lamina occupies the region between the x-axis and the curve $y = x^2 - 4$. Find its centre of mass.

The region is below the axis, $-2 \le x \le 2$, symmetric about the y-axis: $\bar{x} = 0$. Top $y_1 = 0$, bottom $y_2 = x^2 - 4$.

symmetry; top is the axis

$A = \int_{-2}^{2} (4 - x^2)\,dx = 32/3$ — top − bottom



picture: G below the axis

$\int_{-2}^{2} \frac{1}{2}(0 - (x^2 - 4)^2)\,dx = -\frac{1}{2} \times 512/15 = -256/15$, $\bar{y} = -8/5$. G $(0, -1.6)$

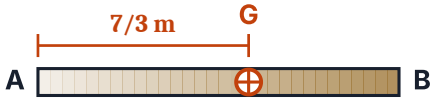
½(y₁² − y₂²): negative

9 A rod AB of length 4 m has mass per unit length $(2 + x)$ kg m^{-1} at a distance x m from A. Find its mass and the distance of its centre of mass from A.

Mass $= \int_0^4 (2 + x)\,dx = 8 + 8 = 16$ kg — mass = ∫λ dx

Moment $= \int_0^4 (2x + x^2)\,dx = 16 + 64/3 = 112/3$

moment = ∫xλ dx



picture: G nearer B

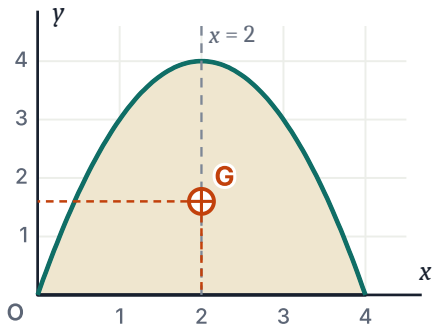
$\bar{x} = (112/3) \div 16 = 7/3 = 2.33$ m from A — ÷ mass

C · Reasoning No calculator. Explain in words.

10 A uniform lamina is bounded by $y = 4x - x^2$ and the x-axis. Without integrating, write down $\bar{x}$. Explain your answer.

$4x - x^2 = x(4 - x)$ is zero at $x = 0$ and 4, and the curve is symmetric about $x = 2$.

symmetry of the curve



picture: symmetric about x = 2

Strips at $2 - t$ and $2 + t$ are equal, so $\bar{x} = 2$.

equal strips either side

11 A uniform lamina fills the region under a curve $y = f(x)$ ($f(x) \geq 0$), and the greatest value of $f(x)$ is 6. Explain why $\bar{y} \leq 3$. When is $\bar{y} = 3$?

Every strip has its centre at $\frac{1}{2}y \leq 3$. G is a weighted mean of these heights, so $\bar{y} \leq 3$.

mean of the strip centres

$\bar{y} = 3$ only if every strip is 6 high: the lamina is a rectangle.

equal only for a rectangle

12 A uniform lamina R lies under a curve with area 3, $\int xy\,dx = 4$ and $\int \frac{1}{2}y^2\,dx = 2.4$. A second lamina is R stretched by scale factor 2 parallel to the y-axis (every y doubled). Find the centre of mass of each.

R: $\bar{x} = 4/3$, $\bar{y} = 2.4 \div 3 = 0.8$ ÷ A

Stretched: A and $\int xy\,dx$ both double, so $\bar{x} = 4/3$ again; $\int \frac{1}{2}y^2\,dx$ is × 4: $\bar{y} = 9.6 \div 6 = 1.6$

y doubled: ȳ doubles

D · Show that Keep the density ρ in a proof. Show every integral.

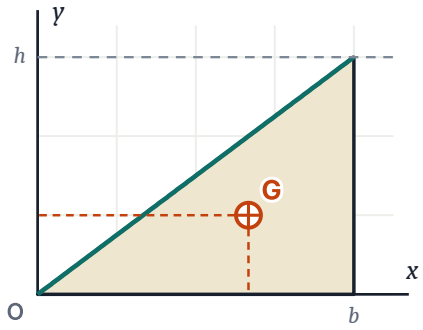
13 A uniform triangular lamina of density ρ is bounded by the line $y = hx/b$, the x-axis and the line $x = b$. Show by integration that its centre of mass is at $(\frac{2}{3}b, \frac{1}{3}h)$.

Mass $= \rho\int_0^b hx/b\,dx = \frac{1}{2}\rho bh$ mass = ρ∫y dx

Moment about Oy: $\rho\int_0^b hx^2/b\,dx = \frac{1}{3}\rho hb^2$,

$\bar{x} = \frac{1}{3}\rho hb^2 \div \frac{1}{2}\rho bh = \frac{2}{3}b$

ρ∫xy dx ÷ mass



picture: (⅔b, ⅓h)

$\rho\int_0^b \frac{1}{2}h^2x^2/b^2\,dx = \frac{1}{6}\rho h^2b$, $\bar{y} = \frac{1}{6}\rho h^2b \div \frac{1}{2}\rho bh = \frac{1}{3}h$

ρ∫½y² dx ÷ mass

14 A uniform lamina is bounded by $y = x^n$ ($n > 0$), the x-axis and the line $x = 1$. Show that $\bar{x} = \dfrac{n+1}{n+2}$. What happens to $\bar{x}$ as n gets large? Explain.

$A = \int_0^1 x^n\,dx = 1/(n+1)$, $\int_0^1 x^{n+1}\,dx = 1/(n+2)$ power rule

$\bar{x} = \dfrac{1/(n+2)}{1/(n+1)} = \dfrac{n+1}{n+2}$ ÷ A

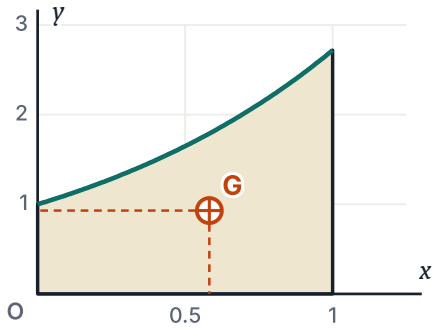
As $n \to \infty$, $\bar{x} \to 1$: the area crowds towards $x = 1$ (the curve is almost flat, then rises steeply).

large n: x̄ → 1

15 A uniform lamina is bounded by $y = e^x$, the coordinate axes and the line $x = 1$. Show that $\bar{x} = \dfrac{1}{e-1}$ and $\bar{y} = \dfrac{e+1}{4}$.

$A = \int_0^1 e^x\,dx = e - 1$ — area

By parts: $\int_0^1 xe^x\,dx = \left[xe^x - e^x\right]_0^1 = 1$, $\bar{x} = 1/(e-1)$ — integration by parts



picture: G

$\int_0^1 \tfrac{1}{2}e^{2x}\,dx = \tfrac{1}{4}(e^2 - 1)$,
$\bar{y} = \tfrac{1}{4}(e-1)(e+1) \div (e-1) = (e+1)/4$ — difference of two squares

E · Exam-style question Show your method: marks are for working.

16 A uniform lamina occupies the region between the curve $y = 6x - x^2$ and the line $y = 2x$. The line meets the curve at O and at A.

(a) Show that the area of the lamina is $32/3$. [2 marks]

(b) Explain why $\bar{x} = 2$. [1 mark]

(c) Show that $\bar{y} = 28/5$. [3 marks]

(d) The lamina hangs freely from O. Find the angle between OA and the vertical. [2 marks]

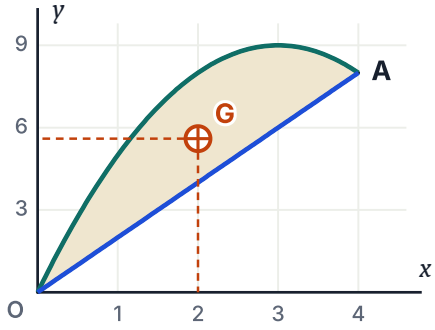
(a) $6x - x^2 = 2x \Rightarrow x = 0, 4$; $A = \int_0^4 (4x - x^2)\,dx = 32 - 64/3 = 32/3$ **M1 A1** — limits, top − bottom

(b) The strip height $4x - x^2 = x(4-x)$ is symmetric about $x = 2$, so $\bar{x} = 2$ **B1** — symmetry of the strips

(c) $\int_0^4 \tfrac{1}{2}((6x - x^2)^2 - 4x^2)\,dx = \int_0^4 \tfrac{1}{2}(32x^2 - 12x^3 + x^4)\,dx$ **M1** — $\tfrac{1}{2}(y_1^2 - y_2^2)$

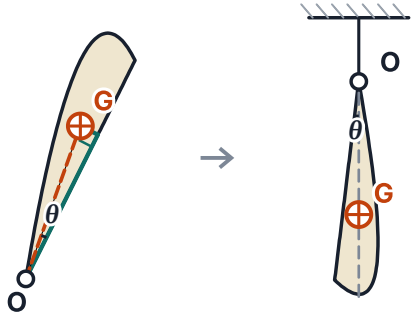
$= \tfrac{1}{2}(2048/3 - 768 + 1024/5) = 896/15$ **A1** — integrate

$\bar{y} = (896/15) \div (32/3) = 28/5$ **A1** — ÷ A



picture: G

(d) A is $(4, 8)$. OA makes $\tan^{-1}(8/4) = 63.43°$ with the x-axis, OG makes $\tan^{-1}(5.6/2) = 70.35°$. **M1** — angle of OA and of OG



picture: θ = 6.9° (hanging from O)

OG is vertical, so the angle between OA and the vertical is $70.35° - 63.43° = 6.9°$ **A1** — difference