Solids of Revolution by Integration

Specification: AQA 7367: ME4 (solids of revolution by integration) · Edexcel 9FM0: FM2 Further centres of mass (uniform and non-uniform solids of revolution)

INDEPENDENT

You do

A · By integration $V = \int \pi y^2\,dx$, then $\bar{x} = \int \pi x y^2\,dx \div V$. Square y first.

1 The region under $y = 2x$ from $x = 0$ to $x = 3$ is rotated about the x-axis. Find $\bar{x}$ of the solid cone.

$y^2 = 4x^2$: $V = \int_0^3 4\pi x^2\,dx = 36\pi$ — square y; volume



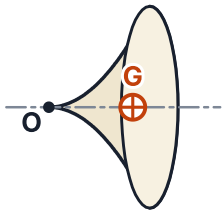
picture: G, ¾ of the way from O

$\int_0^3 4\pi x^3\,dx = 81\pi$, $\bar{x} = 81\pi \div 36\pi = 9/4$ $(= \frac{3}{4}h$ ✓$)$

$\int \pi xy^2\,dx \div V$

2 The region under $y = x^2$ from $x = 0$ to $x = 1$ is rotated about the x-axis. Find $\bar{x}$.

$V = \int_0^1 \pi x^4\,dx = \pi/5$ — $y^2 = x^4$: volume



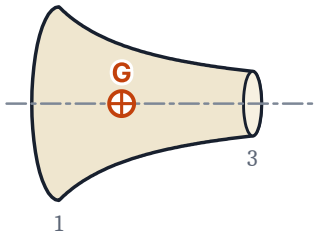
picture: G

$\int_0^1 \pi x^5\,dx = \pi/6$, $\bar{x} = (\pi/6) \div (\pi/5) = 5/6$

$\int \pi xy^2\,dx \div V$

3 The region under $y = 1/x$ from $x = 1$ to $x = 3$ is rotated about the x-axis. Find $\bar{x}$ exactly.

$V = \int_1^3 \pi x^{-2}\,dx = \pi\left[-x^{-1}\right]_1^3 = 2\pi/3$ — $y^2 = x^{-2}$: volume



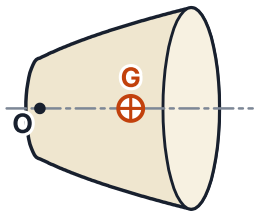
picture: G

$\int_1^3 \pi x^{-1}\,dx = \pi \ln 3$, $\bar{x} = \pi \ln 3 \div (2\pi/3) = (3/2)\ln 3 = 1.65$

$\int 1/x\,dx = \ln x$; $\div V$

4 The region under $y = \sqrt{(x + 1)}$ from $x = 0$ to $x = 3$ is rotated about the x-axis. Find $\bar{x}$.

$y^2 = x + 1$: $V = \int_0^3 \pi(x + 1)\,dx = \pi(9/2 + 3) = 15\pi/2$ — volume



picture: G

$\int_0^3 \pi(x^2 + x)\,dx = \pi(9 + 9/2) = 27\pi/2$, $\bar{x} = 27/15 = 9/5$

$\int \pi xy^2\,dx \div V$

B · Frustums, composites, hanging and density

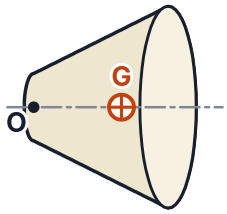
Composite: each part's V and $\bar{x}$, then $\Sigma Vx \div \Sigma V$. Hanging from a rim: $\tan\theta = r \div d$.

5 The region under $y = 1 + \frac{1}{2}x$ from $x = 0$ to $x = 4$ is rotated about the x-axis to form a frustum. Find the distance of its centre of mass from the smaller face.

$y^2 = 1 + x + \frac{1}{4}x^2$: $V = \pi(4 + 8 + 16/3) = 52\pi/3$

expand y^2; volume

$\int_0^4 \pi(x + x^2 + \frac{1}{4}x^3)\,dx = \pi(8 + 64/3 + 16) = 136\pi/3$ $\int \pi xy^2\,dx$



picture: G

$\bar{x} = 136/52 = 34/13 = 2.62$ from the smaller face

$\div$ V

6 The solid of Explore Q1 ($V = 8\pi$, $\bar{x} = 8/3$) has a uniform cylinder of radius 2 and length 2 fixed to its plane face: it runs from $x = 0$ to $x = 6$.
(a) Find $\bar{x}$.
(b) The solid is freely suspended from a point on the rim of its end face at $x = 6$. Find the angle between its axis and the vertical.

(a) Cylinder: $V = 8\pi$ at $x = 5$

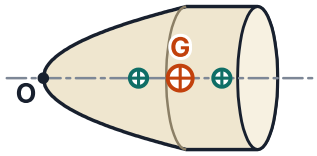
cylinder: V and centre

	V	x	Vx
paraboloid	8π	$8/3$	$64\pi/3$
cylinder	8π	5	40π
Σ	$\mathbf{16\pi}$		$\mathbf{184\pi/3}$

table: V, x, Vx

$\bar{x} = (184\pi/3) \div 16\pi = 23/6 = 3.83$

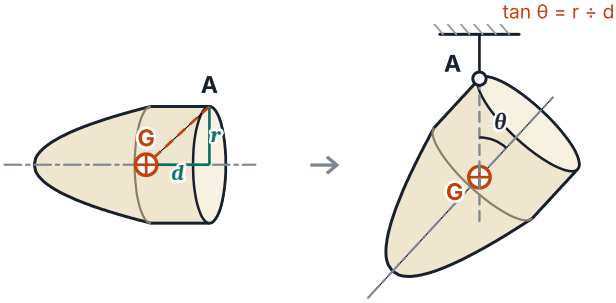
$\Sigma Vx \div \Sigma V$



picture: G

(b) $d = 6 - 23/6 = 13/6$, $r = 2$:
$\tan\theta = 2 \div (13/6) = 12/13$, $\theta = 42.7°$

$\tan\theta = r \div d$

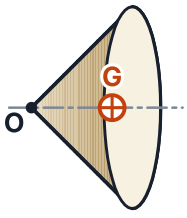


picture: θ = 42.7° (hanging from the rim)

7 The solid cone formed by rotating $y = x$, $0 \le x \le 3$, about the x-axis has density $\rho = kx$, where x is the distance from the vertex O. Find the distance of its centre of mass from O.

Mass $= \pi k\int_0^3 x \cdot x^2\,dx = \pi k\left[\frac{1}{4}x^4\right]_0^3 = 81\pi k/4$ mass = $\int \pi\rho y^2\,dx$

Moment $= \pi k\int_0^3 x^4\,dx = 243\pi k/5$ moment = $\int \pi\rho xy^2\,dx$



picture: G nearer the denser base

$\bar{x} = (243/5) \div (81/4) = 12/5 = 2.4$ from O (uniform: 2.25)

$\div$ mass

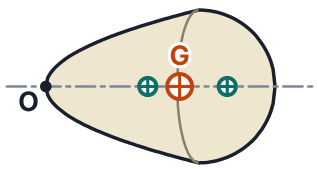
8 The solid of Explore Q1 ($V = 8\pi$, $\bar{x} = 8/3$) has a uniform hemisphere of radius 2 fixed to its plane face: it runs from $x = 0$ to $x = 6$. Find $\bar{x}$.

Hemisphere: $V = 16\pi/3$, centre $4 + 3/4 = 19/4$

hemisphere: 3r/8 beyond the face

	V	x	Vx
paraboloid	8π	$8/3$	$64\pi/3$
hemisphere	$16\pi/3$	$19/4$	$76\pi/3$
Σ	$\mathbf{40\pi/3}$		$\mathbf{140\pi/3}$

table: V, x, Vx



picture: G

$\bar{x} = 140/40 = 7/2 = 3.5$

$\Sigma Vx \div \Sigma V$

9 The region under $y = x^2$ from $x = 0$ to $x = a$ is rotated about the x-axis. The centre of mass of the solid is 5 from O. Find a.

$V = \int_0^a \pi x^4 \, dx = \pi a^5/5$, $\int_0^a \pi x^5 \, dx = \pi a^6/6$

V and $\int \pi xy^2 \, dx$ in terms of a

$\bar{x} = 5a/6 = 5$, so $a = 6$

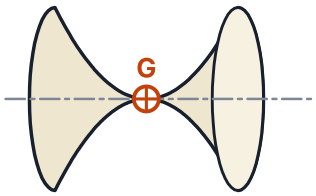
solve $\bar{x} = 5$

C · Reasoning No calculator. Explain in words.

10 The region under $y = x^2$ from $x = -1$ to $x = 1$ is rotated about the x-axis. Without integrating, write down $\bar{x}$. Explain.

$y^2 = x^4$ takes the same value at x and $-x$: discs at $\pm x$ are equal.

equal discs either side



picture: symmetric about x = 0

The solid is symmetric about the plane $x = 0$, so $\bar{x} = 0$.

symmetry

11 A pupil says: "Stretching the region by scale factor 3 parallel to the y-axis moves the centre of mass of the solid of revolution." True or false? Explain.

Every y^2 is multiplied by 9, so $\int \pi y^2 \, dx$ and $\int \pi xy^2 \, dx$ are both × 9.

both integrals × 9

The ratio is unchanged: $\bar{x}$ stays the same (and $\bar{y} = 0$). False.

ratio unchanged

12 A solid cone and a triangular lamina have the same cross-section through the axis (height h). The cone's G is $h/4$ from its base; the triangle's is $h/3$. Explain why the cone's G is nearer the base.

A strip of the lamina has mass $\propto y$; a disc of the cone has mass $\propto y^2$.

strips $\propto$ y, discs $\propto$ y^2

Near the base y is large, so squaring gives the base end even more of the mass: G moves towards the base.

wide end weighs more

D · Show that Keep ρ and π in a proof. Show every integral.

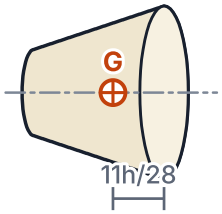
13 A uniform frustum of density ρ is formed by rotating $y = r(1 + x/h)$, $0 \le x \le h$, about the x-axis (radii r and $2r$). Show that its centre of mass is $\frac{11h}{28}$ from its larger face.

Let $t = x/h$ ($dx = h\,dt$): mass
$= \rho\pi r^2 h \int_0^1 (1+t)^2\,dt = \rho\pi r^2 h \times 7/3$

substitute t = x/h; mass

Moment $= \rho\pi r^2 h^2 \int_0^1 t(1+t)^2\,dt = \rho\pi r^2 h^2 (\tfrac{1}{2} + \tfrac{2}{3} + \tfrac{1}{4})$
$= \rho\pi r^2 h^2 \times 17/12$

moment = ρπ∫xy² dx



picture: G, 11h/28 from the larger face

$\bar{x} = (17/12)h \div (7/3) = 17h/28$ from the smaller face, so $h - 17h/28 = 11h/28$ from the larger face

÷ mass; from the other end

14 The region under $y = x^n$ ($n > 0$) from $x = 0$ to $x = 1$ is rotated about the x-axis. Show that $\bar{x} = \frac{2n+1}{2n+2}$. Check your result for $n = 1$ (a cone).

$V = \int_0^1 \pi x^{2n}\,dx = \pi/(2n+1)$, $\int_0^1 \pi x^{2n+1}\,dx = \pi/(2n+2)$

power rule

$\bar{x} = \dfrac{\pi/(2n+2)}{\pi/(2n+1)} = \dfrac{2n+1}{2n+2}$

÷ V

$n = 1$: $\bar{x} = 3/4$, i.e. $3h/4$ from the vertex ✓

check: cone

15 The region under $y = e^x$ from $x = 0$ to $x = 1$ is rotated about the x-axis. Show that $\bar{x} = \frac{e^2+1}{2(e^2-1)}$.

$V = \int_0^1 \pi e^{2x}\,dx = \tfrac{1}{2}\pi(e^2 - 1)$

volume

By parts: $\int_0^1 \pi x e^{2x}\,dx = \pi\left[\tfrac{1}{2}xe^{2x} - \tfrac{1}{4}e^{2x}\right]_0^1 = \tfrac{1}{4}\pi(e^2+1)$

integration by parts



picture: G

$\bar{x} = \tfrac{1}{4}(e^2+1) \div \tfrac{1}{2}(e^2-1) = \dfrac{e^2+1}{2(e^2-1)} = 0.657$

÷ V

E · Exam-style question Show your method: marks are for working.

16 The region bounded by the curve $y^2 = 3x$ ($y \geq 0$), the x-axis and the line $x = 3$ is rotated through 2π about the x-axis to form a uniform solid paraboloid P with its vertex at O.

(a) Show that the volume of P is $27\pi/2$. [2 marks]

(b) Show that the centre of mass of P is 2 from O. [2 marks]

(c) A uniform cylinder of the same material, radius 3 and length 2, is fixed to the plane face of P, so the combined solid runs from $x = 0$ to $x = 5$. Find the distance of its centre of mass from O. [2 marks]

(d) The combined solid hangs freely from a point A on the rim of the cylinder's free end. Find the angle between its axis and the vertical. [2 marks]

(a) $V = \int_0^3 3\pi x \, dx$ **M1** — volume: $\pi\int y^2\,dx$

$= \pi\left[(3/2)x^2\right]_0^3 = 27\pi/2$ **A1** — evaluate

(b) $\int_0^3 3\pi x^2 \, dx = \pi\left[x^3\right]_0^3 = 27\pi$ **M1** — $\int \pi x y^2\,dx$

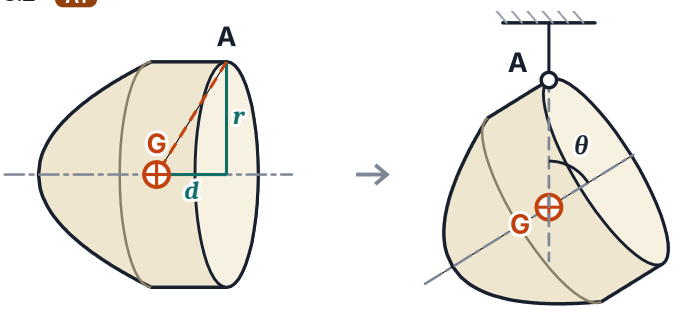
$\bar{x} = 27\pi \div (27\pi/2) = 2$ **A1** — $\div V$

(c) Cylinder: $V = 9\pi \times 2 = 18\pi$ at $x = 4$, $Vx = 72\pi$; P: $Vx = (27\pi/2) \times 2 = 27\pi$: $\bar{x} = (27\pi + 72\pi) \div (27\pi/2 + 18\pi)$ **M1** — $\Sigma Vx \div \Sigma V$

$= 99\pi \div (63\pi/2) = 22/7 = 3.14$ (close to π, by chance) **A1** — simplify

(d) The free end is at $x = 5$: $d = 5 - 22/7 = 13/7$, $r = 3$: $\tan\theta = 3 \div (13/7) = 21/13$ **M1** — $\tan\theta = r \div d$

$\theta = 58.2°$ **A1** — 1 d.p.



picture: θ = 58.2° (hanging from A)