Unit Review

Specification: AQA 7367: ME3, ME4, ME5, ME6 · Edexcel 9FM0: FM2 Further centres of mass (ME6 content, moments and rigid bodies, is AQA only)

INDEPENDENT Mixed practice

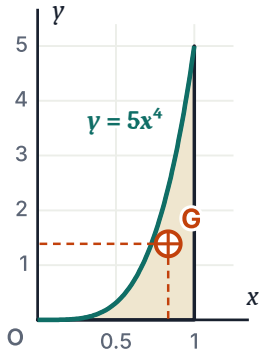
A · Fluency One tool each. Which one?

1 A uniform lamina is bounded by the curve $y = 5x^4$, the x-axis and the line $x = 1$. Find its centre of mass.

$A = \int_0^1 5x^4\,dx = 1$ from L1 — area

$\int_0^1 5x^5\,dx = 5/6$, $\int_0^1 ½ \times 25x^8\,dx = 25/18$ — $\int xy\,dx$; $\int ½y^2\,dx$

$\bar{x} = 5/6$, $\bar{y} = 25/18$ (dividing by $A = 1$) — ÷ A



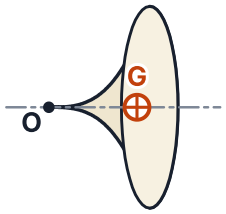
picture: G near the steep end

2 The region bounded by the curve $y = x^3$, the x-axis and the line $x = 1$ is rotated through 360° about the x-axis to form a uniform solid. Find $\bar{x}$.

$V = \pi\int_0^1 x^6\,dx = \pi/7$ from L2 — volume: $\pi\int y^2\,dx$

$\pi\int_0^1 x^7\,dx = \pi/8$ — $\pi\int xy^2\,dx$

$\bar{x} = (\pi/8) \div (\pi/7) = 7/8 = 0.875$ — ÷ V



picture: G nearer the wide end

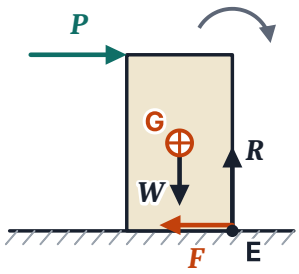
3 A uniform block, 0.9 m wide and 1.5 m tall, of weight 120 N, stands on rough horizontal ground ($\mu = 0.35$). A horizontal force P acts at the middle of the top edge of its back face, at right angles to that face. Does it slide or topple first, and at what value of P?

Slides at $\mu R = 0.35 \times 120 = 42$ N from L3 — limiting friction

Topples: moments about the front edge E:

$P \times 1.5 = 120 \times 0.45$, $P = 36$ N — moments about E

36 < 42: it topples first, at 36 N — smaller force first



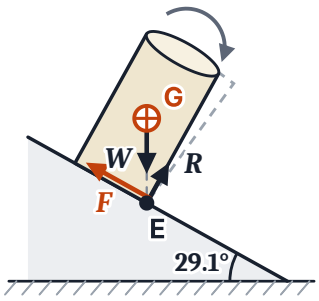
picture: topples about E

4 A uniform solid cylinder of radius 0.25 m and height 0.9 m stands on a plane face on a rough plane ($\mu = 0.6$) whose angle is slowly increased. Does it slide or topple first, and at what angle?

Slides when $\tan\alpha = 0.6$, $\alpha = 31.0°$ from L4 — $\tan\alpha = \mu$

Topples when $\tan\alpha = 0.25/0.45 = 5/9$, $\alpha = 29.1°$ — $\tan\alpha = d/h$

29.1° < 31.0°: it topples first, at 29.1° — smaller angle first



picture: the vertical through G passes through E

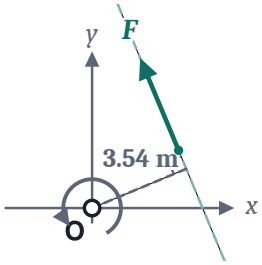
5 A force $(-5\mathbf{i} + 12\mathbf{j})$ N acts at the point $(3, 2)$ (metres).
(a) Find its moment about O.
(b) Find the perpendicular distance from O to its line of action.

(a) $x F_y - y F_x = 3 \times 12 - 2 \times (-5) = 46$: 46 N m anticlockwise from L5

k-component of r × F

(b) $|F| = \sqrt{(5^2 + 12^2)} = 13$; $d = 46/13 = 3.54$ m

moment ÷ magnitude



picture: d = 3.54 m; anticlockwise

6 A uniform ladder AB, of length 4 m and mass 18 kg, rests with A on rough horizontal ground and B against a smooth vertical wall, at 60° to the ground. A child of mass 30 kg stands on the ladder 3 m from A. Find the least possible coefficient of friction at the ground. Take $g = 9.8$ m s^{-2}.

↑: $R = 48g = 470.4$. Moments about A: $S \times 4 \sin 60°$ $= (18g \times 2 + 30g \times 3) \cos 60°$ from L6

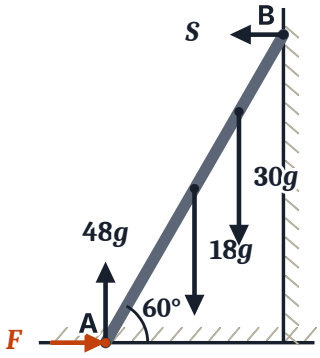
resolve ↑; moments about A

$S = 63g/(4 \sin 60°) = 178.2$ N

solve

$F = S$, $F \le \mu R$: $\mu \ge 178.2/470.4 = 0.379$

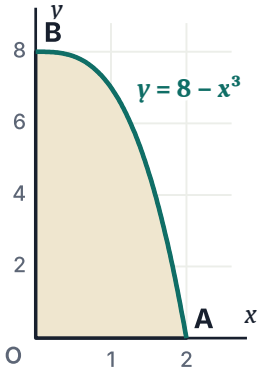
F ≤ μR



picture: friction limiting

B · Multi-step Two or more tools in one question. Draw it first.

7 A uniform lamina OAB is bounded by the curve $y = 8 - x^3$ and the coordinate axes, with O at the origin, A$(2, 0)$ and B$(0, 8)$ (metres).
(a) Find its centre of mass.
The lamina stands in a vertical plane with OA along a line of greatest slope of a rough plane ($\mu = 0.3$), whose angle is slowly increased.
(b) A is the lower end of OA. Does it slide or topple first, and at what angle?
(c) The lamina is turned round so that O is the lower end. Does it slide or topple first, and at what angle?



(a) $A = \int_0^2 (8 - x^3)\,dx = 12$; $\int_0^2 x(8 - x^3)\,dx = 48/5$: $\bar{x} = 4/5$ from L1, L4

area; $\int xy\,dx \div A$

$\int_0^2 \frac{1}{2}(8 - x^3)^2\,dx = \frac{1}{2}\left[64x - 4x^4 + x^7/7\right]_0^2 = 288/7$: $\bar{y} = (288/7) \div 12 = 24/7$

$\int \frac{1}{2}y^2\,dx \div A$

(b) Slides: $\tan\alpha = 0.3$, $\alpha = 16.7°$. Topples about A: $d = 2 - 4/5 = 6/5$, $h = 24/7$: $\tan\alpha = 7/20$, $\alpha = 19.3°$

tan α = μ; tan α = d/h

It slides first, at 16.7°

smaller angle first

(c) Topples about O: $d = 4/5$: $\tan\alpha = (4/5) \div (24/7) = 7/30$, $\alpha = 13.1° < 16.7°$: it topples first, at 13.1°

tan α = d/h; compare

8 A uniform lamina OABC is bounded by the curve $y = 2 + \frac{1}{2}x^2$, the coordinate axes and the line $x = 2$, with O$(0, 0)$, A$(2, 0)$, B$(2, 4)$ and C$(0, 2)$ (metres). It has weight W and stands in a vertical plane with OA on rough horizontal ground ($\mu = 0.25$).
(a) Show that $\bar{x} = 9/8$.
(b) A horizontal force P acts at B in the direction OA, and is slowly increased. Does the lamina slide or topple first?
(c) Instead, a horizontal force acts at B in the direction AO. Does it slide or topple first?

(a) $A = \int_0^2 (2 + \frac{1}{2}x^2)\,dx = 4 + 4/3 = 16/3$; $\int_0^2 (2x + \frac{1}{2}x^3)\,dx = 4 + 2 = 6$ from L1, L3

area; $\int xy\,dx$

$\bar{x} = 6 \div (16/3) = 9/8$

÷ A

(b) Topples about A: $d = 2 - 9/8 = 7/8$: $4P = 7W/8$, $P = 7W/32 = 0.219W$; slides at $0.25W$

moments about A; μW

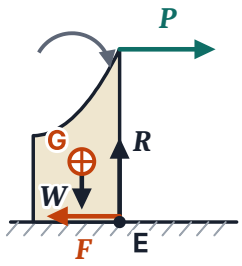
$7W/32 < W/4$: it topples first.

smaller force first

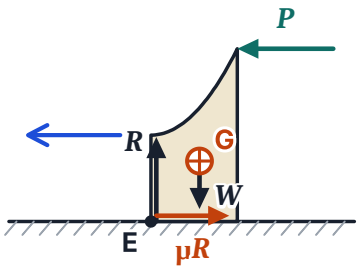
(c) Topples about O: $d = 9/8$: $4P = 9W/8$, $P = 9W/32 = 0.281W > 0.25W$: it slides first, when $P = 0.25W$

moments about O; compare

(b)

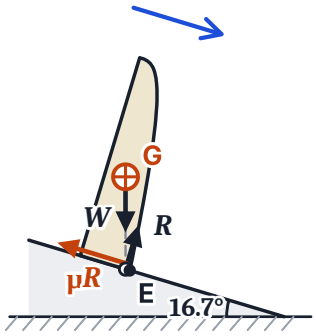


(c)

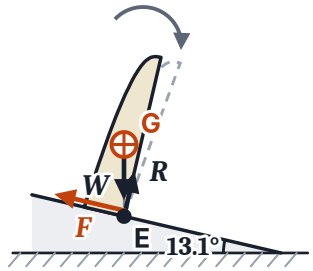


picture: (b) topples about A, (c) slides

(b)



(c)



picture: (b) slides, (c) topples about O

9 A square lamina OABC has side 4 m, with O at the origin, A on the positive x-axis and C on the positive y-axis. Forces of 3 N along AO, 5 N along BC, 6 N along OC and 2 N along AB act on it, in the directions given by the letters.
(a) Find the magnitude of the resultant and where its line of action crosses OA.
(b) A single force is added at C so that the system reduces to a couple. Find this force and the moment of the couple.

(a) $R = (-3-5)\mathbf{i} + (6+2)\mathbf{j}$: $R = -8\mathbf{i} + 8\mathbf{j}$, $|R| = 8\sqrt{2} = 11.3$ N from L5

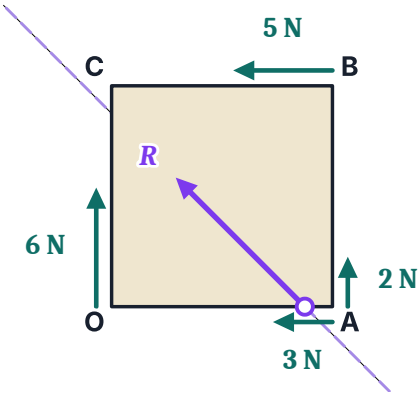
add the components

Moments about O: BC $4 \times 5 = 20$, AB $4 \times 2 = 8$, AO and OC 0: $M_O = 28$ anticlockwise

moments about O

Line of action: $x\,R_y - y\,R_x = M_O$: $8x + 8y = 28$; on OA ($y = 0$): $x = 3.5$ m from O

the line of action



picture: R crosses OA 3.5 m from O

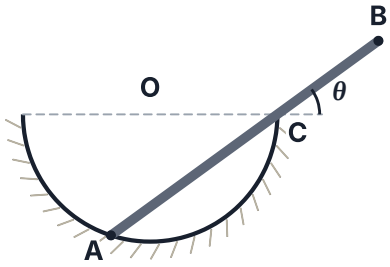
(b) A couple has zero resultant: add $(8\mathbf{i} - 8\mathbf{j})$ N at $C(0, 4)$. Its moment about O: $0 \times (-8) - 4 \times 8 = -32$

$\Sigma F = 0$

Couple: $28 - 32 = -4$: 4 N m clockwise

total moment

10 A smooth hemispherical bowl, of radius 0.5 m and centre O, is fixed with its rim horizontal. A uniform rod AB, of length 1.3 m and weight 36 N, rests with A on the inside of the bowl and the rod touching the rim at C, at θ to the horizontal.
(a) Show that $AC = \cos\theta$.
(b) Find θ.
(c) Find the reactions at A and at C.



(a) OA = OC = 0.5 and angle OCA = θ: $AC = 2 \times 0.5 \cos\theta = \cos\theta$ from L6

isosceles triangle OAC

(b) At A the reaction is along AO; at C it is $\perp$ the rod.
About A: $R_C \cos\theta = 36 \times 0.65 \cos\theta$, $R_C = 23.4$ N

moments about A

Along the rod: $R_A \cos\theta = 36 \sin\theta$; $\perp$ the rod: $R_A \sin\theta + 23.4 = 36\cos\theta$

AO makes θ with the rod

With $c = \cos\theta$: $36(1 - c^2) + 23.4c = 36c^2$, $72c^2 - 23.4c - 36 = 0$

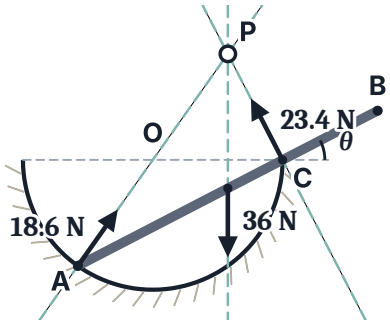
eliminate; quadratic in cos θ

$c = 0.8880$, $\theta = 27.4°$

positive root

(c) $R_A = 36 \tan\theta = 18.6$ N; $R_C = 23.4$ N

reaction at A = W tan θ



picture: the three forces meet at P

11 A uniform block, 0.6 m wide and 1.0 m tall, of weight 200 N, stands on rough horizontal ground ($\mu = 0.3$).

(a) A couple acts on the block in the vertical plane through G that is perpendicular to its front face. Explain why the block cannot slide, and find the greatest couple for which it does not tip.

(b) Instead, a couple of 30 N m acts in the same plane, in the sense that would tip the block about its front edge E, and a horizontal force P acts at the middle of the top edge of its back face, towards E. P is slowly increased. Does the block slide or topple first, and at what value of P?

(a) A couple has zero resultant force, so no friction is needed: the block cannot slide from L3, L5

a couple: no resultant force

It tips about an edge when the couple exceeds the moment of the weight about that edge: $200 \times 0.3 = 60$: greatest couple 60 N m

moments about E

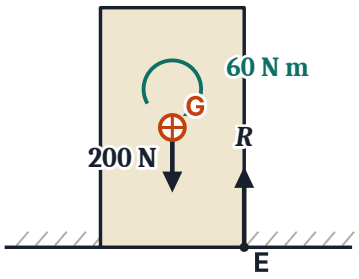
(b) Slides at $\mu R = 0.3 \times 200 = 60$ N limiting friction

Topples: moments about E: $1.0P + 30 = 200 \times 0.3$, $P = 30$ N

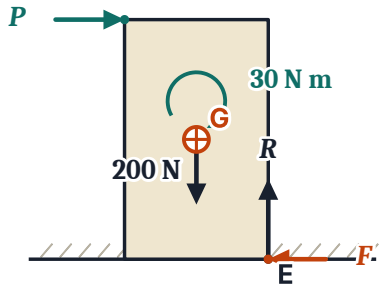
moments about E: P and the couple

$30 < 60$: it topples first, at 30 N smaller force first

(a)



(b)



picture: (a) tipping, (b) P and the couple

C · Reasoning No calculator. Explain in words.

12 The region bounded by $y = x^2$, the x-axis and the line $x = 1$ is used twice: as a uniform lamina, and rotated about the x-axis to make a uniform solid. Sam says: "They are made from the same region, so they have the same $\bar{x}$." Show that Sam is wrong and explain why.

Lamina: $\bar{x} = \int_0^1 x^3\,dx \div \int_0^1 x^2\,dx = (1/4) \div (1/3) = 3/4$

from L1, L2

strips: $\int xy\,dx \div \int y\,dx$

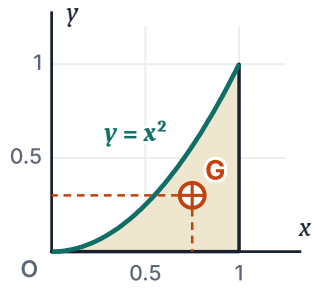
Solid: $\bar{x} = \int_0^1 \pi x^5\,dx \div \int_0^1 \pi x^4\,dx = (1/6) \div (1/5) = 5/6$

discs: $\int \pi xy^2\,dx \div \int \pi y^2\,dx$

$5/6 \neq 3/4$. A strip's mass is proportional to y but a disc's to y^2, so the solid gives more weight to the wide end.

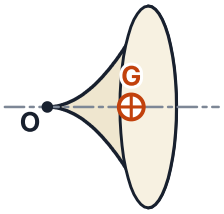
compare

lamina



picture: the two centres of mass

solid



13 Two uniform blocks have the same size and shape, but one weighs twice as much as the other. Each stands on the same rough horizontal ground, and a horizontal force acts on each at the same height, at the middle of its back face. Priya says: "The heavier block is harder to push, so it is more likely to topple before it slides." Is she right? Explain.

To slide: $P = \mu W$. To topple: $Ph = Wd$, so $P = Wd/h$ from L3

the two forces

Both are proportional to W: doubling W doubles both. Which comes first depends only on μ against d/h.

the weight cancels

No: both blocks do the same thing first (they slide first if $\mu < d/h$).

compare

14 A rigid body is in equilibrium under three forces, two of which are parallel. Explain why the third force must be parallel to them too.

Resolve at right angles to the two parallel forces: they have no component in that direction from L5, L6

resolve ⊥ the parallel pair

The components must add to zero, so the third force has no component in that direction either: it is parallel to the other two.

$\Sigma F = 0$

Taking moments then fixes where its line of action lies.

moments fix its line

D · Show that Keep letters; simplify at the end.

15 The region bounded by the curve $y = r\sqrt{(x/h)}$, the x-axis and the line $x = h$ is rotated through 360° about the x-axis to form a uniform solid paraboloid with vertex O, height h and base radius r.

(a) Show that its centre of mass is $\frac{2h}{3}$ from O.

(b) It stands on its plane face on a rough plane (coefficient of friction μ) whose angle is slowly increased. Show that it topples before it slides if $h > 3r/\mu$.

(c) Given $r = 0.2$ m and $\mu = 0.5$, find the range of heights for which it topples first.

(a) $y^2 = r^2x/h$: $V = \pi\int_0^h r^2x/h\,dx = \pi r^2h/2$,
$\pi\int_0^h r^2x^2/h\,dx = \pi r^2h^2/3$ from L2, L4

$\pi\int y^2\,dx$ and $\pi\int xy^2\,dx$

$\bar{x} = (\pi r^2h^2/3) \div (\pi r^2h/2) = 2h/3$ ÷ V

(b) G is $h - 2h/3 = h/3$ above the plane face. Topples when $\tan\alpha = r \div (h/3) = 3r/h$; slides when $\tan\alpha = \mu$

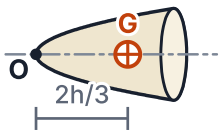
tan α = d/h; tan α = μ

Topples first if $3r/h < \mu$, that is, if $h > 3r/\mu$

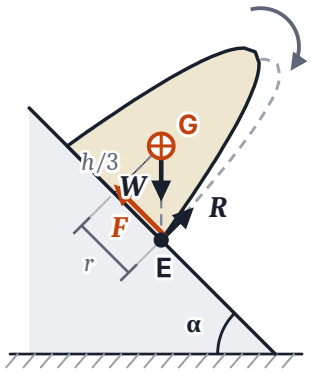
smaller angle first

(c) $h > 3 \times 0.2 \div 0.5 = 1.2$: $h > 1.2$ m substitute

(a)



(b)



picture: G is h/3 above the face

16 A uniform ladder AB, of length $2a$ and weight W, rests with A on rough horizontal ground (coefficient of friction μ) and B against a smooth vertical wall, at angle θ to the ground, where $\tan\theta < 1/(2\mu)$. A horizontal force P acts at A, towards the wall.

(a) Show that the least value of P for equilibrium is $\frac{W(1 - 2\mu\tan\theta)}{2\tan\theta}$.

(b) Find this least force when $W = 300$ N, $\theta = 50°$ and $\mu = 0.3$.

(a) ↑: $R = W$. Moments about A:
$S \times 2a\sin\theta = W \times a\cos\theta$, $S = W/(2\tan\theta)$ from L6

resolve ↑; moments about A

For the least P, A is about to slip away from the wall: friction acts towards the wall and is limiting, $F = \mu W$

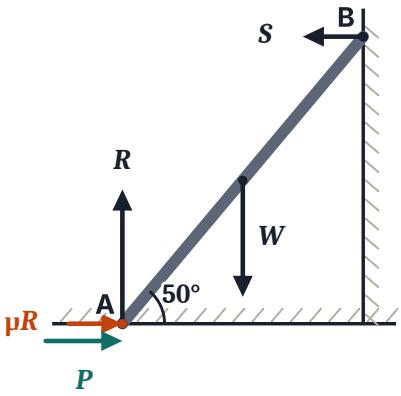
limiting friction: its direction

→: $P + \mu W = S$: $P = W/(2\tan\theta) - \mu W = \frac{W(1 - 2\mu\tan\theta)}{2\tan\theta}$

resolve →

(b) $P = 300(1 - 0.6\tan 50°)/(2\tan 50°) = 35.9$ N

substitute



picture: P and μR both towards the wall

E · Exam-style questions Show your method: marks are for working.

17 *AQA-style*. A uniform lamina is bounded by the curve $y = x^2(3 - x)$ and the x-axis, between O$(0, 0)$ and A$(3, 0)$ (metres). Its weight is 60 N.

(a) Show that the centre of mass of the lamina is 1.8 m from the y-axis. [3]

The lamina is freely hinged at O to a fixed point and rests in a vertical plane with OA horizontal and the lamina above OA. It is held by a light string attached at A and to the fixed point D, 4 m vertically above the point $(6, 0)$.

(b) Find the tension in the string. [3]

(c) Find the magnitude and direction of the force at the hinge. [2]

(a) $A = \int_0^3 (3x^2 - x^3)\,dx = \left[x^3 - \tfrac{1}{4}x^4\right]_0^3 = 27 - 81/4 = 27/4$ from L1, L6 **B1** — area

$\int_0^3 (3x^3 - x^4)\,dx = \left[\tfrac{3}{4}x^4 - \tfrac{1}{5}x^5\right]_0^3 = 243/4 - 243/5 = 243/20$ **M1** — $\int xy\,dx$

$\bar{x} = (243/20) \div (27/4) = 9/5 = 1.8$ m **A1** — $\div$ A

(b) $AD = \sqrt{(3^2 + 4^2)} = 5$: the string pulls along AD, so its vertical component is $4T/5$, acting 3 m from O **B1** — components of T

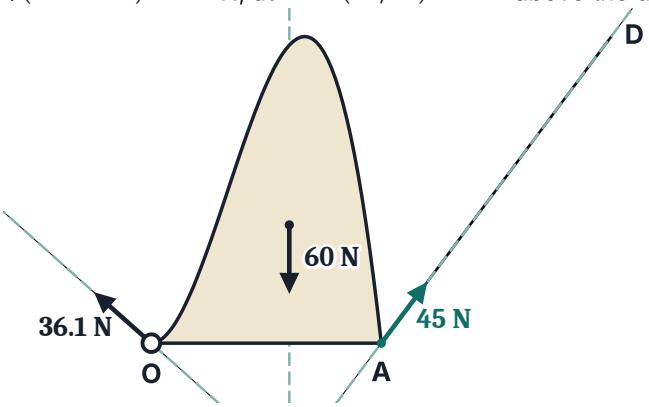
Moments about O: $3 \times 4T/5 = 60 \times 1.8$, $2.4T = 108$ **M1** — moments about O

$T = 45$ N **A1** — solve

(c) The string pulls $45 \times 3/5 = 27$ N in the direction OA, so the hinge pushes 27 N in the direction AO; ↑:

$Y = 60 - 45 \times 4/5 = 24$ **M1** — resolve twice

$\sqrt{(27^2 + 24^2)} = 36.1$ N, at $\tan^{-1}(24/27) = 41.6°$ above the direction AO **A1** — magnitude and direction



picture: the three forces meet

18 *Edexcel-style*. The region bounded by the curve $y = \sqrt[3]{x}$, the x-axis and the line $x = 8$ is rotated through $360°$ about the x-axis to form a uniform solid S. Lengths are in centimetres.

(a) Show that the centre of mass of S is 5 cm from O. [4]

S stands with its plane face on a rough plane whose angle is slowly increased. The coefficient of friction is 0.6.

(b) Determine whether S slides or topples first, and find the angle at which this happens. [3]

(c) Find the range of values of the coefficient of friction for which S would topple first. [1]

(a) $y^2 = x^{2/3}$: $V = \pi\int_0^8 x^{2/3}\,dx = \pi\left[\tfrac{3}{5}x^{5/3}\right]_0^8 = 96\pi/5$ from L2, L4 **M1 A1**

volume: $\pi\int y^2\,dx$

$\pi\int_0^8 x^{5/3}\,dx = \pi\left[\tfrac{3}{8}x^{8/3}\right]_0^8 = 96\pi$ **M1**

$\pi\int xy^2\,dx$

$\bar{x} = 96\pi \div (96\pi/5) = 5$ cm from O **A1**

÷ V

(b) G is $8 - 5 = 3$ cm above the plane face, whose radius is $\sqrt[3]{8} = 2$ cm. Topples when $\tan\alpha = 2/3$, $\alpha = 33.7°$ **M1**

$\tan\alpha = d/h$

Slides when $\tan\alpha = 0.6$, $\alpha = 31.0°$ **M1**

$\tan\alpha = \mu$

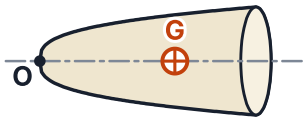
$31.0° < 33.7°$: S slides first, at $31.0°$ **A1**

smaller angle first

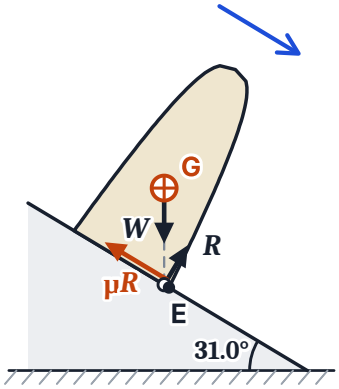
(c) Topples first if $\mu > 2/3$ **B1**

$\mu > d/h$

(a)



(b)



picture: (a) G, (b) about to slide