The Scalar Product

Specification: AQA 7366/7367: F3, F4 · Edexcel 9FM0: Core Pure 6.3, 6.4

INDEPENDENT **You do**

A · Scalar products Multiply matching components, then add.

1 Find $\begin{pmatrix}3\\1\\4\end{pmatrix}\cdot\begin{pmatrix}2\\-5\\1\end{pmatrix}$.

$3\times 2+1\times(-5)+4\times 1$ — multiply matching components

$=6-5+4=5$ — add

2 Find $\begin{pmatrix}-2\\4\\3\end{pmatrix}\cdot\begin{pmatrix}5\\1\\-2\end{pmatrix}$.

$(-2)\times 5+4\times 1+3\times(-2)$ — multiply matching components

$=-10+4-6=-12$ — add

3 Find $(2\mathbf{i}-3\mathbf{j}+\mathbf{k})\cdot(4\mathbf{i}+\mathbf{j}-2\mathbf{k})$.

$\begin{pmatrix}2\\-3\\1\end{pmatrix}\cdot\begin{pmatrix}4\\1\\-2\end{pmatrix}$ — i, j, k → columns

$2\times 4+(-3)\times 1+1\times(-2)$ — multiply matching components

$=8-3-2=3$ — add

4 Find $(5\mathbf{i}-2\mathbf{k})\cdot(3\mathbf{i}+7\mathbf{j}+4\mathbf{k})$.

$\begin{pmatrix}5\\0\\-2\end{pmatrix}\cdot\begin{pmatrix}3\\7\\4\end{pmatrix}$ — i, j, k → columns

$5\times 3+0\times 7+(-2)\times 4$ — multiply matching components

$=15+0-8=7$ — add

B · Angles between vectors $\cos\theta = \mathbf{a}\cdot\mathbf{b} \div |\mathbf{a}||\mathbf{b}|$

5 Find the angle between $\mathbf{a}=\begin{pmatrix}1\\2\\2\end{pmatrix}$ and $\mathbf{b}=\begin{pmatrix}2\\-1\\2\end{pmatrix}$.

$\mathbf{a}\cdot\mathbf{b}=1\times 2+2\times(-1)+2\times 2$ — multiply matching components

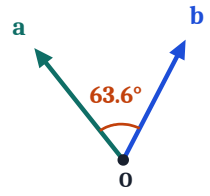
$=2-2+4=4$ — add

$|\mathbf{a}|=\sqrt{(1^2+2^2+2^2)}=3$ — magnitude

$|\mathbf{b}|=\sqrt{(2^2+(-1)^2+2^2)}=3$ — magnitude

$\cos\theta=\dfrac{4}{3\times 3}$ — $\cos\theta = \mathbf{a}\cdot\mathbf{b} \div |\mathbf{a}||\mathbf{b}|$

$\theta=63.6°$ — inverse cos, 1 d.p.



picture: acute angle

6 Find the angle between $\mathbf{a}=\begin{pmatrix}3\\0\\4\end{pmatrix}$ and $\mathbf{b}=\begin{pmatrix}0\\5\\12\end{pmatrix}$.

$\mathbf{a}\cdot\mathbf{b}=3\times 0+0\times 5+4\times 12$ — multiply matching components

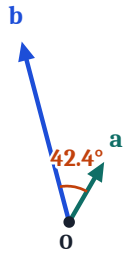
$=0+0+48=48$ — add

$|\mathbf{a}|=\sqrt{(3^2+0^2+4^2)}=5$ — magnitude

$|\mathbf{b}|=\sqrt{(0^2+5^2+12^2)}=13$ — magnitude

$\cos\theta=\dfrac{48}{5\times 13}$ — $\cos\theta = \mathbf{a}\cdot\mathbf{b} \div |\mathbf{a}||\mathbf{b}|$

$\theta=42.4°$ — inverse cos, 1 d.p.



picture: acute angle

7 Find the angle between $\mathbf{a}=\begin{pmatrix}2\\-3\\6\end{pmatrix}$ and $\mathbf{b}=\begin{pmatrix}-1\\4\\8\end{pmatrix}$.

$\mathbf{a}\cdot\mathbf{b}=2\times(-1)+(-3)\times 4+6\times 8$ — multiply matching components

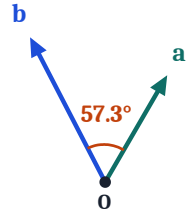
$=-2-12+48=34$ — add

$|\mathbf{a}|=\sqrt{(2^2+(-3)^2+6^2)}=7$ — magnitude

$|\mathbf{b}|=\sqrt{((-1)^2+4^2+8^2)}=9$ — magnitude

$\cos\theta=\dfrac{34}{7\times 9}$ — $\cos\theta = \mathbf{a}\cdot\mathbf{b} \div |\mathbf{a}||\mathbf{b}|$

$\theta=57.3°$ — inverse cos, 1 d.p.



picture: acute angle

8 Find the angle between $\mathbf{a}=4\mathbf{i}-\mathbf{j}+\mathbf{k}$ and $\mathbf{b}=-2\mathbf{i}+3\mathbf{j}+\mathbf{k}$.

$\begin{pmatrix}4\\-1\\1\end{pmatrix},\begin{pmatrix}-2\\3\\1\end{pmatrix}$ — i, j, k → columns

$\mathbf{a}\cdot\mathbf{b}=4\times(-2)+(-1)\times 3+1\times 1$ — multiply matching components

$=-8-3+1=-10$ — add

$|\mathbf{a}|=\sqrt{(4^2+(-1)^2+1^2)}=\sqrt{18}$ — magnitude

$|\mathbf{b}|=\sqrt{((-2)^2+3^2+1^2)}=\sqrt{14}$ — magnitude

$\cos\theta=\dfrac{-10}{\sqrt{18}\times\sqrt{14}}$ — $\cos\theta = \mathbf{a}\cdot\mathbf{b} \div |\mathbf{a}||\mathbf{b}|$

$\theta=129.0°$ — inverse cos, 1 d.p.



picture: obtuse angle

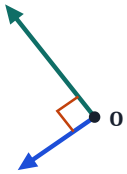
C · Reasoning No calculator. Explain in words as well as numbers.

9 Determine whether $\begin{pmatrix}6\\-2\\5\end{pmatrix}$ and $\begin{pmatrix}3\\4\\-2\end{pmatrix}$ are perpendicular.

$\begin{pmatrix}6\\-2\\5\end{pmatrix}\cdot\begin{pmatrix}3\\4\\-2\end{pmatrix} = 6\times3 + (-2)\times4 + 5\times(-2)$ — multiply matching components

$= 18 - 8 - 10 = 0$ — add

The scalar product is 0, so they are perpendicular. — a·b = 0: perpendicular



picture: right angle

10 Without finding the angle, decide whether the angle between $\begin{pmatrix}1\\-3\\4\end{pmatrix}$ and $\begin{pmatrix}2\\5\\3\end{pmatrix}$ is acute, a right angle or obtuse.

$\mathbf{a}\cdot\mathbf{b} = 1\times2 + (-3)\times5 + 4\times3$ — multiply matching components

$= 2 - 15 + 12 = -1$ — add

$\mathbf{a}\cdot\mathbf{b} < 0 \Rightarrow \cos\theta < 0$, so the angle is obtuse. — negative: obtuse

11 $|\mathbf{a}| = 4$, $|\mathbf{b}| = 5$ and the angle between $\mathbf{a}$ and $\mathbf{b}$ is 60°. Find $\mathbf{a}\cdot\mathbf{b}$ and $\mathbf{a}\cdot\mathbf{a}$.

$\mathbf{a}\cdot\mathbf{b} = 4\times5\times\cos 60° = 10$ — a·b = |a||b| cos θ

$\mathbf{a}\cdot\mathbf{a} = |\mathbf{a}|^2 = 16$ — a·a = |a|²

12 Jo says that two vectors with $|\mathbf{a}| = 2$ and $|\mathbf{b}| = 5$ have $\mathbf{a}\cdot\mathbf{b} = 12$. Explain why Jo must be wrong.

$\mathbf{a}\cdot\mathbf{b} = 2\times5\times\cos\theta = 10\cos\theta$ — a·b = |a||b| cos θ

$\cos\theta \le 1$, so $\mathbf{a}\cdot\mathbf{b} \le 10$. It cannot be 12. — |cos θ| ≤ 1

D · Algebra and "show that" Set the scalar product equal to 0.

13 Find k so that $\begin{pmatrix}k\\3\\-2\end{pmatrix}$ and $\begin{pmatrix}4\\k\\6\end{pmatrix}$ are perpendicular.

$4k + 3k - 12 = 0$ — a·b = 0

$7k = 12 \Rightarrow k = \frac{12}{7}$ — solve

14 Find the possible values of p for which $\begin{pmatrix}p\\4\\1\end{pmatrix}$ and $\begin{pmatrix}p\\p\\-12\end{pmatrix}$ are perpendicular.

$p^2 + 4p - 12 = 0$ — a·b = 0

$(p+6)(p-2) = 0$ — factorise

$p = 2$ or $p = -6$ — both values

15 Find a vector of the form $\begin{pmatrix}1\\p\\q\end{pmatrix}$ that is perpendicular to both $\begin{pmatrix}1\\1\\1\end{pmatrix}$ and $\begin{pmatrix}1\\2\\3\end{pmatrix}$.

$1 + p + q = 0$ — perpendicular to the first

$1 + 2p + 3q = 0$ — perpendicular to the second

$q = -1 - p \Rightarrow 1 + 2p - 3 - 3p = 0$ — substitute

$p = -2, q = 1$ — solve

$\begin{pmatrix}1\\-2\\1\end{pmatrix}$ — the vector

16 The points A(2, 1, −1), B(4, 3, 0) and C(5, 1, 2) form a triangle. Show that the triangle has a right angle at B.

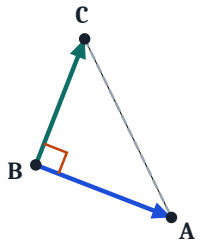
$\overrightarrow{BA} = \begin{pmatrix}2\\1\\-1\end{pmatrix} - \begin{pmatrix}4\\3\\0\end{pmatrix} = \begin{pmatrix}-2\\-2\\-1\end{pmatrix}$ — both vectors start at B

$\overrightarrow{BC} = \begin{pmatrix}5\\1\\2\end{pmatrix} - \begin{pmatrix}4\\3\\0\end{pmatrix} = \begin{pmatrix}1\\-2\\2\end{pmatrix}$ — BC = c − b

$\overrightarrow{BA}\cdot\overrightarrow{BC} = (-2)\times1 + (-2)\times(-2) + (-1)\times2$ — multiply matching components

$= -2 + 4 - 2 = 0$ — add

The scalar product is 0, so angle ABC = 90°. — a·b = 0: perpendicular



picture: right angle at B

E · Exam-style question Show your method: marks are for working.

17 The points A, B and C have coordinates $(2, 0, 1)$, $(4, 3, -5)$ and $(5, -2, k)$ respectively, where k is a constant.

(a) Find $\overrightarrow{AB} \cdot \overrightarrow{AC}$ in terms of k. [2 marks]

(b) Angle BAC is a right angle. Find the value of k. [2 marks]

(c) Using your value of k, find angle ABC, giving your answer to the nearest 0.1°. [4 marks]

(a) $\overrightarrow{AB} = \begin{pmatrix} 2 \\ 3 \\ -6 \end{pmatrix}, \overrightarrow{AC} = \begin{pmatrix} 3 \\ -2 \\ k-1 \end{pmatrix}$ **M1** — vectors from A

$\overrightarrow{AB} \cdot \overrightarrow{AC} = 6 - 6 - 6(k-1) = 6 - 6k$ **A1** — multiply matching components

(b) $6 - 6k = 0$ **M1** — right angle: a·b = 0

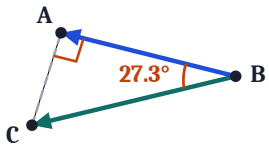
$k = 1$ **A1** — solve

(c) $\overrightarrow{BA} = \begin{pmatrix} -2 \\ -3 \\ 6 \end{pmatrix}, \overrightarrow{BC} = \begin{pmatrix} 1 \\ -5 \\ 6 \end{pmatrix}$ **M1** — both vectors start at B

$\overrightarrow{BA} \cdot \overrightarrow{BC} = -2 + 15 + 36 = 49, |\overrightarrow{BA}| = 7, |\overrightarrow{BC}| = \sqrt{62}$ **M1** — scalar product and magnitudes

$\cos ABC = \dfrac{49}{7\sqrt{62}}$ **A1** — cos θ = a·b ÷ |a||b|

angle ABC = 27.3° **A1** — inverse cos, 0.1°



picture: angle at B