Vector Equation of a Line

Specification: AQA 7366/7367: F1 · Edexcel 9FM0: Core Pure 6.1

INDEPENDENT ## You do

A · Writing equations r = a + λd. Through two points: d = b − a.

1 Find a vector equation of the line through $(3, -2, 1)$ in the direction $\begin{pmatrix} 4 \\ 1 \\ -5 \end{pmatrix}$.

$\mathbf{r} = \begin{pmatrix} 3 \\ -2 \\ 1 \end{pmatrix} + \lambda \begin{pmatrix} 4 \\ 1 \\ -5 \end{pmatrix}$ — r = a + λd

2 Find a vector equation of the line through $(0, 6, -1)$ in the direction $2\mathbf{i} - 3\mathbf{k}$.

$2\mathbf{i} - 3\mathbf{k} = \begin{pmatrix} 2 \\ 0 \\ -3 \end{pmatrix}$ — missing j: write 0

$\mathbf{r} = \begin{pmatrix} 0 \\ 6 \\ -1 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ 0 \\ -3 \end{pmatrix}$ — r = a + λd

3 Find a vector equation of the line through A(1, −3, 2) and B(4, 0, −1).

$\overrightarrow{AB} = \begin{pmatrix} 4 \\ 0 \\ -1 \end{pmatrix} - \begin{pmatrix} 1 \\ -3 \\ 2 \end{pmatrix} = \begin{pmatrix} 3 \\ 3 \\ -3 \end{pmatrix}$ — direction: $\overrightarrow{AB}$ = **b** − **a**

$\begin{pmatrix} 3 \\ 3 \\ -3 \end{pmatrix} = 3\begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}$, so use $\begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}$ as the direction — simplify: ÷ 3

$\mathbf{r} = \begin{pmatrix} 1 \\ -3 \\ 2 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}$ — r = a + λd

4 Find a vector equation of the line through P(−2, 5, 0) and Q(1, 5, 4).

$\overrightarrow{PQ} = \begin{pmatrix} 1 \\ 5 \\ 4 \end{pmatrix} - \begin{pmatrix} -2 \\ 5 \\ 0 \end{pmatrix} = \begin{pmatrix} 3 \\ 0 \\ 4 \end{pmatrix}$ — direction: $\overrightarrow{PQ}$ = **q** − **p**

$\mathbf{r} = \begin{pmatrix} -2 \\ 5 \\ 0 \end{pmatrix} + \lambda \begin{pmatrix} 3 \\ 0 \\ 4 \end{pmatrix}$ — r = a + λd

B · Points on lines Find λ from one component. Check the other two.

5 Find the coordinates of the point on the line $\mathbf{r} = \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 4 \\ -2 \end{pmatrix}$ where λ = −3.

$\mathbf{r} = \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix} - 3\begin{pmatrix} 1 \\ 4 \\ -2 \end{pmatrix} = \begin{pmatrix} -1 \\ -13 \\ 9 \end{pmatrix}$ — substitute λ = −3

The point $(-1, -13, 9)$ — read off the point

6 Show that the point $(6, 7, -5)$ lies on the line $\mathbf{r} = \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 2 \\ -2 \end{pmatrix}$.

x: $2 + \lambda = 6 \Rightarrow \lambda = 4$ — x: find λ

y: $-1 + 2\lambda = 7 \Rightarrow \lambda = 4$ ✓ — y: same λ

z: $3 - 2\lambda = -5 \Rightarrow \lambda = 4$ ✓ — z: same λ

All three components agree (λ = 4), so $(6, 7, -5)$ lies on the line. — all agree: on the line

7 Determine whether the point $(-1, 5, 8)$ lies on the line $\mathbf{r} = \begin{pmatrix} 5 \\ -1 \\ 2 \end{pmatrix} + \mu \begin{pmatrix} -3 \\ 3 \\ 2 \end{pmatrix}$.

x: $5 - 3\mu = -1 \Rightarrow \mu = 2$ — x: find μ

y: $-1 + 3\mu = 5 \Rightarrow \mu = 2$ ✓ — y: same μ

z: $2 + 2\mu = 8 \Rightarrow \mu = 3$, not 2 — z: different μ

The components do not agree, so $(-1, 5, 8)$ does **not** lie on the line. — disagree: not on the line

8 The point $(p, 1, q)$ lies on the line $\mathbf{r} = \begin{pmatrix} 3 \\ -5 \\ 2 \end{pmatrix} + t\begin{pmatrix} 2 \\ 3 \\ -1 \end{pmatrix}$.

Find p and q.

y: $-5 + 3t = 1 \Rightarrow t = 2$ — use the known component

x: $p = 3 + 2 \times 2 = 7$ — substitute t = 2

z: $q = 2 - 2 = 0$ — substitute t = 2

$p = 7, q = 0$ — both values

C · Reasoning No calculator. Explain in words as well as numbers.

9 Show that $\mathbf{r} = \begin{pmatrix}1\\2\\-1\end{pmatrix} + \lambda\begin{pmatrix}2\\-1\\3\end{pmatrix}$ and $\mathbf{r} = \begin{pmatrix}5\\0\\5\end{pmatrix} + \mu\begin{pmatrix}-4\\2\\-6\end{pmatrix}$ are the same line.

$\begin{pmatrix}-4\\2\\-6\end{pmatrix} = -2\begin{pmatrix}2\\-1\\3\end{pmatrix}$, so the lines are parallel. — compare the directions

x: $1 + 2\lambda = 5 \Rightarrow \lambda = 2$ — x: find λ

y: $2 - \lambda = 0 \Rightarrow \lambda = 2$ ✓ — y: same λ

z: $-1 + 3\lambda = 5 \Rightarrow \lambda = 2$ ✓ — z: same λ

$(5, 0, 5)$ is on both lines. Parallel with a common point: the same line.

parallel + common point: same



picture: one line

10 The points A$(1, 1, 2)$ and B$(3, 5, -4)$ are given. Show that the line AB is parallel to the line $\mathbf{r} = \begin{pmatrix}0\\2\\7\end{pmatrix} + \lambda\begin{pmatrix}-1\\-2\\3\end{pmatrix}$.

$\overrightarrow{AB} = \begin{pmatrix}3\\5\\-4\end{pmatrix} - \begin{pmatrix}1\\1\\2\end{pmatrix} = \begin{pmatrix}2\\4\\-6\end{pmatrix}$ — direction of AB

$\begin{pmatrix}2\\4\\-6\end{pmatrix} = -2\begin{pmatrix}-1\\-2\\3\end{pmatrix}$ — compare the directions

The direction vectors are parallel, so the lines are parallel.

multiple ⇒ parallel



picture: parallel lines

11 Show that the line $\mathbf{r} = \begin{pmatrix}3\\0\\-1\end{pmatrix} + \lambda\begin{pmatrix}2\\-1\\4\end{pmatrix}$ is perpendicular to the vector $\begin{pmatrix}1\\6\\1\end{pmatrix}$.

$\begin{pmatrix}2\\-1\\4\end{pmatrix} \cdot \begin{pmatrix}1\\6\\1\end{pmatrix} = 2 \times 1 + (-1) \times 6 + 4 \times 1$ — multiply matching components

$= 2 - 6 + 4 = 0$ — add

The scalar product is 0: the line's direction is perpendicular to the vector.

scalar product 0: perpendicular

12 The points A$(1, 2, 3)$, B$(3, -1, 5)$ and C$(7, k, 9)$ lie on a straight line. Find k.

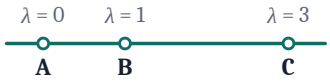
$\overrightarrow{AB} = \begin{pmatrix}3\\-1\\5\end{pmatrix} - \begin{pmatrix}1\\2\\3\end{pmatrix} = \begin{pmatrix}2\\-3\\2\end{pmatrix}$ — direction: $\overrightarrow{AB}$ = **b** − **a**

$\mathbf{r} = \begin{pmatrix}1\\2\\3\end{pmatrix} + \lambda\begin{pmatrix}2\\-3\\2\end{pmatrix}$ — r = a + λd

x: $1 + 2\lambda = 7 \Rightarrow \lambda = 3$ — C on line AB: find λ

z: $3 + 2 \times 3 = 9$ ✓ — check z

y: $k = 2 - 3 \times 3 = -7$ — y gives k



picture: C at λ = 3

D · Algebra and "show that" Parallel lines share a direction. A distance is a magnitude.

13 Find a vector equation of the line through $(4, -3, 1)$ that is parallel to the line $\mathbf{r} = \begin{pmatrix}1\\1\\0\end{pmatrix} + \mu\begin{pmatrix}2\\-5\\3\end{pmatrix}$.

Parallel, so use the same direction $\begin{pmatrix}2\\-5\\3\end{pmatrix}$ — parallel: same direction

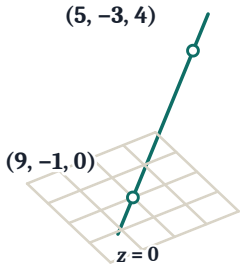
$\mathbf{r} = \begin{pmatrix}4\\-3\\1\end{pmatrix} + \lambda\begin{pmatrix}2\\-5\\3\end{pmatrix}$ — r = a + λd

14 Find the coordinates of the point where the line $\mathbf{r} = \begin{pmatrix}5\\-3\\4\end{pmatrix} + \lambda\begin{pmatrix}2\\1\\-2\end{pmatrix}$ crosses the plane $z = 0$.

z: $4 - 2\lambda = 0 \Rightarrow \lambda = 2$ — set z = 0

$\mathbf{r} = \begin{pmatrix}5\\-3\\4\end{pmatrix} + 2\begin{pmatrix}2\\1\\-2\end{pmatrix} = \begin{pmatrix}9\\-1\\0\end{pmatrix}$ — substitute λ = 2

The point $(9, -1, 0)$ — read off the point



picture: the line meets z = 0

15 The line l passes through $A(2, -1, 5)$ and $B(6, 1, -1)$. The point C lies on l and B is the midpoint of AC. Find the coordinates of C.

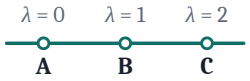
$\overrightarrow{AB} = \begin{pmatrix}6\\1\\-1\end{pmatrix} - \begin{pmatrix}2\\-1\\5\end{pmatrix} = \begin{pmatrix}4\\2\\-6\end{pmatrix}$ — direction: AB = b − a

$\mathbf{r} = \begin{pmatrix}2\\-1\\5\end{pmatrix} + \lambda\begin{pmatrix}4\\2\\-6\end{pmatrix}$: $\lambda = 0$ at A, $\lambda = 1$ at B — keep AB: $\lambda = 1$ at B

B is the midpoint of AC, so C has $\lambda = 2$ — twice as far: $\lambda = 2$

$\mathbf{r} = \begin{pmatrix}2\\-1\\5\end{pmatrix} + 2\begin{pmatrix}4\\2\\-6\end{pmatrix} = \begin{pmatrix}10\\3\\-7\end{pmatrix}$ — substitute $\lambda = 2$

The point $(10, 3, -7)$ — read off the point



picture: B halfway from A to C

16 The line l has equation $\mathbf{r} = \begin{pmatrix}1\\2\\-1\end{pmatrix} + \lambda\begin{pmatrix}2\\-1\\2\end{pmatrix}$ and passes through $A(1, 2, -1)$. Find the coordinates of the two points on l that are a distance 6 from A.

$\overrightarrow{AP} = \lambda\begin{pmatrix}2\\-1\\2\end{pmatrix}$ — AP = r − a

$|\overrightarrow{AP}| = |\lambda| \times \sqrt{(2^2 + (-1)^2 + 2^2)} = 3|\lambda|$ — magnitude

$3|\lambda| = 6 \Rightarrow \lambda = 2$ or $\lambda = -2$ — both values

$\lambda = 2$: $\begin{pmatrix}1\\2\\-1\end{pmatrix} + 2\begin{pmatrix}2\\-1\\2\end{pmatrix} = \begin{pmatrix}5\\0\\3\end{pmatrix}$ — substitute $\lambda = 2$

$\lambda = -2$: $\begin{pmatrix}1\\2\\-1\end{pmatrix} - 2\begin{pmatrix}2\\-1\\2\end{pmatrix} = \begin{pmatrix}-3\\4\\-5\end{pmatrix}$ — substitute $\lambda = -2$

The points $(5, 0, 3)$ and $(-3, 4, -5)$ — both points



picture: 6 each side of A

E · Exam-style question Show your method: marks are for working.

17 The points A and B have coordinates $(3, -1, 2)$ and $(5, 3, -4)$ respectively.

(a) Find a vector equation of the line AB. [2 marks]

(b) Show that the point $C(0, -7, 11)$ lies on the line AB. [2 marks]

(c) The line l passes through $D(1, 4, 0)$ and is parallel to AB. Write down a vector equation of l. [1 mark]

(d) Show that l and the line AB are not the same line. [2 marks]

(a) $\overrightarrow{AB} = \begin{pmatrix}2\\4\\-6\end{pmatrix} = 2\begin{pmatrix}1\\2\\-3\end{pmatrix}$ M1 — direction: AB = b − a

$\mathbf{r} = \begin{pmatrix}3\\-1\\2\end{pmatrix} + \lambda\begin{pmatrix}1\\2\\-3\end{pmatrix}$ A1 — r = a + λd

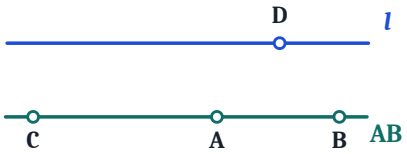
(b) x: $3 + \lambda = 0 \Rightarrow \lambda = -3$ M1 — x: find λ

y: $-1 + 2(-3) = -7$ ✓, z: $2 - 3(-3) = 11$ ✓. All three agree, so C lies on AB. A1 — check y and z

(c) $\mathbf{r} = \begin{pmatrix}1\\4\\0\end{pmatrix} + \mu\begin{pmatrix}1\\2\\-3\end{pmatrix}$ B1 — same direction, through D

(d) Test D on AB: x: $3 + \lambda = 1 \Rightarrow \lambda = -2$ M1 — test D on AB

y: $-1 + 2(-2) = -5 \neq 4$. D is not on AB, so the parallel lines are different. E1 — no common point



picture: l parallel to AB