Cartesian Form of a Line

Specification: AQA 7366/7367: F1 · Edexcel 9FM0: Core Pure 6.1

INDEPENDENT **You do**

A · Vector → Cartesian Make λ the subject of each component. A direction component of 0 gives $x = a_1$.

1 Find a Cartesian equation of the line $\mathbf{r} = \begin{pmatrix}4\\-1\\2\end{pmatrix} + \lambda\begin{pmatrix}3\\2\\5\end{pmatrix}$.

$x = 4 + 3\lambda, y = -1 + 2\lambda, z = 2 + 5\lambda$ — components of **r**

$\lambda = \frac{x-4}{3}, \lambda = \frac{y+1}{2}, \lambda = \frac{z-2}{5}$ — make λ the subject

$\frac{x-4}{3} = \frac{y+1}{2} = \frac{z-2}{5}$ — each equals λ

2 Find a Cartesian equation of the line $\mathbf{r} = \begin{pmatrix}0\\3\\-5\end{pmatrix} + \lambda\begin{pmatrix}-2\\1\\4\end{pmatrix}$.

$x = -2\lambda, y = 3 + \lambda, z = -5 + 4\lambda$ — components of **r**

$\lambda = \frac{x}{-2}, \lambda = y - 3, \lambda = \frac{z+5}{4}$ — make λ the subject

$\frac{x}{-2} = y - 3 = \frac{z+5}{4}$ — each equals λ

3 Find a Cartesian equation of the line $\mathbf{r} = (6\mathbf{i} - 2\mathbf{k}) + \lambda(5\mathbf{i} + 4\mathbf{j} - 3\mathbf{k})$.

$\mathbf{r} = \begin{pmatrix}6\\0\\-2\end{pmatrix} + \lambda\begin{pmatrix}5\\4\\-3\end{pmatrix}$ — i, j, k → columns

$x = 6 + 5\lambda, y = 4\lambda, z = -2 - 3\lambda$ — components of **r**

$\lambda = \frac{x-6}{5}, \lambda = \frac{y}{4}, \lambda = \frac{z+2}{-3}$ — make λ the subject

$\frac{x-6}{5} = \frac{y}{4} = \frac{z+2}{-3}$ — each equals λ

4 Find a Cartesian equation of the line $\mathbf{r} = \begin{pmatrix}1\\-4\\3\end{pmatrix} + \mu\begin{pmatrix}2\\0\\-5\end{pmatrix}$.

$x = 1 + 2\mu, y = -4, z = 3 - 5\mu$ — components of **r**

$\mu = \frac{x-1}{2}, \mu = \frac{z-3}{-5}$; $y = -4$ for every μ — μ the subject; y fixed

$\frac{x-1}{2} = \frac{z-3}{-5}, y = -4$ — $d_2 = 0$: y on its own

B · Cartesian → vector Rewrite each fraction as (x − a)/d first.

5 Find a vector equation of the line $\frac{x-5}{2} = \frac{y+4}{-3} = \frac{z-1}{6}$.

Point $(5, -4, 1)$ (each top is 0 there) — tops = 0: the point

Direction $\begin{pmatrix}2\\-3\\6\end{pmatrix}$ (the denominators) — denominators: direction

$\mathbf{r} = \begin{pmatrix}5\\-4\\1\end{pmatrix} + \lambda\begin{pmatrix}2\\-3\\6\end{pmatrix}$ — r = a + λd

6 Find a vector equation of the line $\frac{x+1}{3} = \frac{4-y}{2} = \frac{z}{-5}$.

$\frac{4-y}{2} = \frac{y-4}{-2}$ — top and bottom × (−1)

$\frac{z}{-5} = \frac{z-0}{-5}$ — z = z − 0

Point $(-1, 4, 0)$ (each top is 0 there) — tops = 0: the point

Direction $\begin{pmatrix}3\\-2\\-5\end{pmatrix}$ (the denominators) — denominators: direction

$\mathbf{r} = \begin{pmatrix}-1\\4\\0\end{pmatrix} + \lambda\begin{pmatrix}3\\-2\\-5\end{pmatrix}$ — r = a + λd

7 Find a vector equation of the line $\frac{3x-2}{6} = \frac{y+5}{4} = \frac{1-z}{3}$.

$\frac{3x-2}{6} = \frac{x-\frac{2}{3}}{2}$ — ÷ top and bottom by 3

$\frac{1-z}{3} = \frac{z-1}{-3}$ — top and bottom × (−1)

Point $(\frac{2}{3}, -5, 1)$ (each top is 0 there) — tops = 0: the point

Direction $\begin{pmatrix}2\\4\\-3\end{pmatrix}$ (the denominators) — denominators: direction

$\mathbf{r} = \begin{pmatrix}\frac{2}{3}\\-5\\1\end{pmatrix} + \lambda\begin{pmatrix}2\\4\\-3\end{pmatrix}$ — r = a + λd

8 Find a vector equation of the line $x = -3, \frac{y-2}{7} = \frac{z+6}{2}$.

Point $(-3, 2, -6)$ (each top is 0 there) — tops = 0: the point

Direction $\begin{pmatrix}0\\7\\2\end{pmatrix}$ (x is fixed, so 0) — denominators; x fixed: 0

$\mathbf{r} = \begin{pmatrix}-3\\2\\-6\end{pmatrix} + \lambda\begin{pmatrix}0\\7\\2\end{pmatrix}$ — r = a + λd

C · Using the Cartesian form

Substitute into every fraction: all must give the same value.

9 Determine whether the point $(7, -1, -8)$ lies on the line $\frac{x-1}{3} = \frac{y+5}{2} = \frac{z+2}{-3}$.

x: $\frac{7-1}{3} = 2$ — x: work out the fraction

y: $\frac{-1+5}{2} = 2$ ✓ — y: same value

z: $\frac{-8+2}{-3} = 2$ ✓ — z: same value

All equal 2 (this is λ): $(7, -1, -8)$ is on the line.

all equal: on the line

10 The point $(p, q, 5)$ lies on the line $\frac{x+2}{4} = \frac{y-3}{-1} = \frac{z-1}{2}$. Find p and q.

z: $\frac{5-1}{2} = 2$ — use the known component

x: $\frac{p+2}{4} = 2 \Rightarrow p = 6$ — x: set equal to 2

y: $\frac{q-3}{-1} = 2 \Rightarrow q = 1$ — y: set equal to 2

$p = 6, q = 1$ — both values

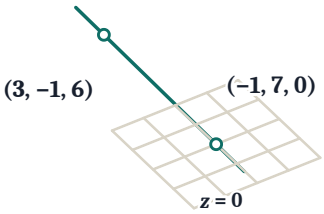
11 Find the coordinates of the point where the line $\frac{x-3}{2} = \frac{y+1}{-4} = \frac{z-6}{3}$ meets the plane $z = 0$.

$z = 0$: $\frac{0-6}{3} = -2$ — set z = 0

$\frac{x-3}{2} = -2 \Rightarrow x = -1$ — x: set equal to −2

$\frac{y+1}{-4} = -2 \Rightarrow y = 7$ — y: set equal to −2

The point $(-1, 7, 0)$ — the point



picture: the line meets z = 0

12 Find a Cartesian equation of the line through A(2, 5, −1) and B(4, 1, 5).

$\overrightarrow{AB} = \begin{pmatrix}4\\1\\5\end{pmatrix} - \begin{pmatrix}2\\5\\-1\end{pmatrix} = \begin{pmatrix}2\\-4\\6\end{pmatrix}$ — direction: $\overrightarrow{AB}$ = **b** − **a**

$\begin{pmatrix}2\\-4\\6\end{pmatrix} = 2\begin{pmatrix}1\\-2\\3\end{pmatrix}$, so use $\begin{pmatrix}1\\-2\\3\end{pmatrix}$ as the direction — simplify: ÷ 2

$\mathbf{r} = \begin{pmatrix}2\\5\\-1\end{pmatrix} + \lambda\begin{pmatrix}1\\-2\\3\end{pmatrix}$ — r = a + λd

$x - 2 = \frac{y-5}{-2} = \frac{z+1}{3}$ — Cartesian: each equals λ

D · Reasoning

No calculator. Parallel lines: directions are multiples. Explain in words as well as numbers.

13 Show that the line $\frac{x-1}{2} = \frac{y+4}{-1} = \frac{z}{3}$ is perpendicular to the vector $4\mathbf{i} + 5\mathbf{j} - \mathbf{k}$ (compare Do Now Q4).

Direction of the line: $\begin{pmatrix}2\\-1\\3\end{pmatrix}$ — denominators: the direction

$\begin{pmatrix}2\\-1\\3\end{pmatrix} \cdot \begin{pmatrix}4\\5\\-1\end{pmatrix} = 2 \times 4 + (-1) \times 5 + 3 \times (-1)$

multiply matching components

$= 8 - 5 - 3 = 0$ — add

The scalar product is 0, so the line is perpendicular to the vector.

a·b = 0: perpendicular

14 Show that l_1: $\frac{x-2}{3} = \frac{y+1}{-1} = \frac{z-4}{2}$ and l_2: $\frac{8-x}{6} = \frac{y+3}{2} = \frac{z-8}{-4}$ are the same line.

$\frac{8-x}{6} = \frac{x-8}{-6}$ — top and bottom × (−1)

$\begin{pmatrix}-6\\2\\-4\end{pmatrix} = -2\begin{pmatrix}3\\-1\\2\end{pmatrix}$, so the lines are parallel. — multiple ⇒ parallel

x: $\frac{8-2}{3} = 2$ — x: work out the fraction

y: $\frac{-3+1}{-1} = 2$ ✓ — y: same value

z: $\frac{8-4}{2} = 2$ ✓ — z: same value

$(8, -3, 8)$ (on l_2) is on l_1. Parallel with a common point: the same line.

parallel + common point: same



picture: one line

15 Find a Cartesian equation of the line through $(-1, 2, 0)$ that is parallel to the line $\frac{x+4}{3} = \frac{1-y}{2}, z = 5$.

$\frac{1-y}{2} = \frac{y-1}{-2}$ — top and bottom × (−1)

$z = 5$ for every point, so the direction is $\begin{pmatrix} 3 \\ -2 \\ 0 \end{pmatrix}$ — z fixed: $d_3 = 0$

$\frac{x+1}{3} = \frac{y-2}{-2}, z = 0$ — same direction, new point

16 The lines $\frac{x-1}{k} = \frac{y+2}{3} = \frac{z-5}{-6}$ and $\frac{x+4}{2} = \frac{y}{-1} = \frac{z-3}{2}$ are parallel. Find k.

Directions $\begin{pmatrix} k \\ 3 \\ -6 \end{pmatrix}$ and $\begin{pmatrix} 2 \\ -1 \\ 2 \end{pmatrix}$ — denominators: the directions

$3 = -3 \times (-1)$ and $-6 = -3 \times 2$: the multiple is -3 — find the multiple

$k = -3 \times 2 = -6$ — same multiple for x

E · Exam-style question Show your method: marks are for working.

17 The line l has Cartesian equation $\frac{x-4}{2} = \frac{1-y}{3} = z + 2$.

(a) Find a vector equation of l. [2 marks]

(b) Find the coordinates of the point where l meets the plane $z = 0$. [2 marks]

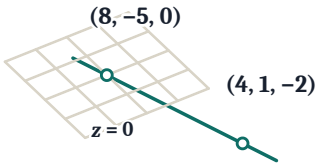
(c) Show that the point $A(0, 7, -4)$ lies on l. [2 marks]

(d) Show that l is perpendicular to the vector $3\mathbf{i} + \mathbf{j} - 3\mathbf{k}$. [2 marks]

(a) $\frac{1-y}{3} = \frac{y-1}{-3}$ and $z + 2 = \frac{z+2}{1}$ M1 — rewrite as (x − a)/d

$\mathbf{r} = \begin{pmatrix} 4 \\ 1 \\ -2 \end{pmatrix} + \lambda \begin{pmatrix} 2 \\ -3 \\ 1 \end{pmatrix}$ A1 — r = a + λd

(b) $z = 0 \Rightarrow \lambda = 2$: $x = 4 + 2(2) = 8, y = 1 - 3(2) = -5$, $(8, -5, 0)$ M1 A1 — set z = 0; substitute



picture: l meets z = 0

(c) $\frac{0-4}{2} = -2, \frac{1-7}{3} = -2, -4 + 2 = -2$ M1 — work out each fraction

All three equal -2, so A lies on l. A1 — all equal: on the line

(d) $2 \times 3 + (-3) \times 1 + 1 \times (-3) = 6 - 3 - 3 = 0$ M1 — direction · vector

The scalar product is 0, so l is perpendicular to $3\mathbf{i} + \mathbf{j} - 3\mathbf{k}$. E1 — a·b = 0: perpendicular