Angle Between Two Lines

Specification: AQA 7366/7367: F1, F4 · Edexcel 9FM0: Core Pure 6.3, 6.4

INDEPENDENT **You do**

A · Lines in vector form Use the direction vectors only. A negative scalar product: take the modulus.

1 Find the acute angle between the lines $l_1: \mathbf{r} = \begin{pmatrix}1\\0\\3\end{pmatrix} + \lambda\begin{pmatrix}2\\2\\-1\end{pmatrix}$ and $l_2: \mathbf{r} = \begin{pmatrix}0\\-2\\1\end{pmatrix} + \mu\begin{pmatrix}6\\-3\\2\end{pmatrix}$.

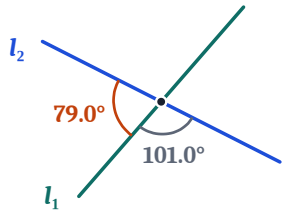
Directions: $\mathbf{d}_1 = \begin{pmatrix}2\\2\\-1\end{pmatrix}, \mathbf{d}_2 = \begin{pmatrix}6\\-3\\2\end{pmatrix}$ — directions only

$\mathbf{d}_1 \cdot \mathbf{d}_2 = 2 \times 6 + 2 \times (-3) + (-1) \times 2 = 12 - 6 - 2 = 4$ — multiply matching components; add

$|\mathbf{d}_1| = \sqrt{(2^2 + 2^2 + (-1)^2)} = 3, |\mathbf{d}_2| = \sqrt{(6^2 + (-3)^2 + 2^2)} = 7$ — magnitudes

$\cos\theta = \frac{4}{3 \times 7}$ — $\cos\theta = |\mathbf{d}_1 \cdot \mathbf{d}_2| \div |\mathbf{d}_1||\mathbf{d}_2|$

$\theta = 79.0°$ — inverse cos, 1 d.p.



picture: the acute angle

2 Find the acute angle between the lines $l_1: \mathbf{r} = \begin{pmatrix}2\\5\\-1\end{pmatrix} + \lambda\begin{pmatrix}4\\-4\\7\end{pmatrix}$ and $l_2: \mathbf{r} = \begin{pmatrix}1\\1\\1\end{pmatrix} + \mu\begin{pmatrix}2\\-1\\-2\end{pmatrix}$.

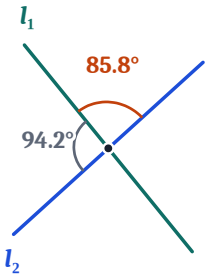
Directions: $\mathbf{d}_1 = \begin{pmatrix}4\\-4\\7\end{pmatrix}, \mathbf{d}_2 = \begin{pmatrix}2\\-1\\-2\end{pmatrix}$ — directions only

$\mathbf{d}_1 \cdot \mathbf{d}_2 = 4 \times 2 + (-4) \times (-1) + 7 \times (-2) = 8 + 4 - 14 = -2$ — multiply matching components; add

$|\mathbf{d}_1| = \sqrt{(4^2 + (-4)^2 + 7^2)} = 9, |\mathbf{d}_2| = \sqrt{(2^2 + (-1)^2 + (-2)^2)} = 3$ — magnitudes

$\cos\theta = \frac{|-2|}{9 \times 3} = \frac{2}{9 \times 3}$ — modulus: the acute angle

$\theta = 85.8°$ — inverse cos, 1 d.p.



picture: the acute angle

3 Find the acute angle between the lines $l_1: \mathbf{r} = (\mathbf{i} - \mathbf{j}) + \lambda(3\mathbf{i} - 4\mathbf{k})$ and $l_2: \mathbf{r} = (2\mathbf{j} + \mathbf{k}) + \mu(\mathbf{i} - 2\mathbf{j} + 2\mathbf{k})$.

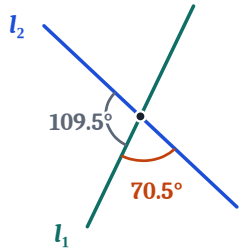
$\mathbf{d}_1 = \begin{pmatrix}3\\0\\-4\end{pmatrix}, \mathbf{d}_2 = \begin{pmatrix}1\\-2\\2\end{pmatrix}$ — i, j, k → columns

$\mathbf{d}_1 \cdot \mathbf{d}_2 = 3 \times 1 + 0 \times (-2) + (-4) \times 2 = 3 + 0 - 8 = -5$ — multiply matching components; add

$|\mathbf{d}_1| = \sqrt{(3^2 + 0^2 + (-4)^2)} = 5, |\mathbf{d}_2| = \sqrt{(1^2 + (-2)^2 + 2^2)} = 3$ — magnitudes

$\cos\theta = \frac{|-5|}{5 \times 3} = \frac{5}{5 \times 3}$ — modulus: the acute angle

$\theta = 70.5°$ — inverse cos, 1 d.p.



picture: the acute angle

4 Find the acute angle between the lines $l_1: \mathbf{r} = \begin{pmatrix}0\\2\\-3\end{pmatrix} + \lambda\begin{pmatrix}1\\3\\-2\end{pmatrix}$ and $l_2: \mathbf{r} = \begin{pmatrix}4\\0\\1\end{pmatrix} + \mu\begin{pmatrix}2\\-1\\4\end{pmatrix}$.

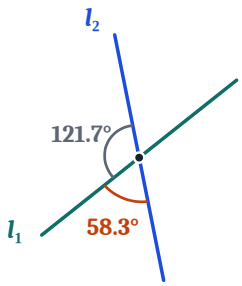
Directions: $\mathbf{d}_1 = \begin{pmatrix}1\\3\\-2\end{pmatrix}, \mathbf{d}_2 = \begin{pmatrix}2\\-1\\4\end{pmatrix}$ — directions only

$\mathbf{d}_1 \cdot \mathbf{d}_2 = 1 \times 2 + 3 \times (-1) + (-2) \times 4 = 2 - 3 - 8 = -9$ — multiply matching components; add

$|\mathbf{d}_1| = \sqrt{(1^2 + 3^2 + (-2)^2)} = \sqrt{14}, |\mathbf{d}_2| = \sqrt{(2^2 + (-1)^2 + 4^2)} = \sqrt{21}$ — magnitudes

$\cos\theta = \frac{|-9|}{\sqrt{14} \times \sqrt{21}} = \frac{9}{\sqrt{14} \times \sqrt{21}}$ — modulus: the acute angle

$\theta = 58.3°$ — inverse cos, 1 d.p.



picture: the acute angle

B · Cartesian and mixed forms Rearrange awkward fractions first, then read the directions (Lesson 3).

5 Find the acute angle between the lines $l_1: \frac{x-2}{3} = \frac{y+1}{-2} = \frac{z}{6}$ and $l_2: x+4 = \frac{y-5}{4} = \frac{z-1}{8}$.

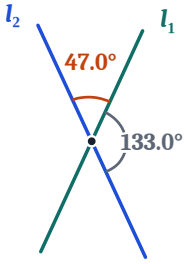
Directions (the denominators): $\mathbf{d}_1 = \begin{pmatrix}3\\-2\\6\end{pmatrix}, \mathbf{d}_2 = \begin{pmatrix}1\\4\\8\end{pmatrix}$ — denominators: directions

$\mathbf{d}_1 \cdot \mathbf{d}_2 = 3\times1 + (-2)\times4 + 6\times8 = 3 - 8 + 48 = 43$ — multiply matching components; add

$|\mathbf{d}_1| = \sqrt{(3^2 + (-2)^2 + 6^2)} = 7, |\mathbf{d}_2| = \sqrt{(1^2 + 4^2 + 8^2)} = 9$ — magnitudes

$\cos\theta = \frac{43}{7\times9}$ — $\cos\theta = |\mathbf{d}_1\cdot\mathbf{d}_2| \div |\mathbf{d}_1||\mathbf{d}_2|$

$\theta = 47.0°$ — inverse cos, 1 d.p.



picture: the acute angle

6 Find the acute angle between the lines $l_1: \frac{1-x}{2} = \frac{y+3}{2} = \frac{2z-1}{2}$ and $l_2: \mathbf{r} = \begin{pmatrix}0\\1\\1\end{pmatrix} + \mu\begin{pmatrix}4\\0\\-3\end{pmatrix}$.

$\frac{1-x}{2} = \frac{x-1}{-2}; \frac{2z-1}{2} = \frac{z-½}{1}$ — rewrite as (x − a)/d

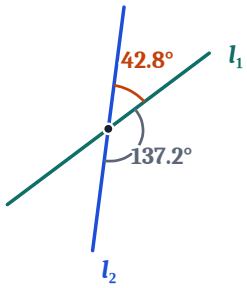
Directions: $\mathbf{d}_1 = \begin{pmatrix}-2\\2\\1\end{pmatrix}, \mathbf{d}_2 = \begin{pmatrix}4\\0\\-3\end{pmatrix}$ — denominators; μ vector

$\mathbf{d}_1 \cdot \mathbf{d}_2 = (-2)\times4 + 2\times0 + 1\times(-3) = -8 + 0 - 3 = -11$ — multiply matching components; add

$|\mathbf{d}_1| = \sqrt{((-2)^2 + 2^2 + 1^2)} = 3, |\mathbf{d}_2| = \sqrt{(4^2 + 0^2 + (-3)^2)} = 5$ — magnitudes

$\cos\theta = \frac{|-11|}{3\times5} = \frac{11}{3\times5}$ — modulus: the acute angle

$\theta = 42.8°$ — inverse cos, 1 d.p.



picture: the acute angle

7 Find the acute angle between the lines $l_1: x = 3, \frac{y-1}{4} = \frac{z+2}{-3}$ and $l_2: x+1 = \frac{y}{2} = \frac{z-2}{2}$.

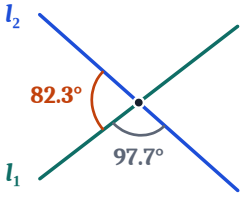
Directions: $\mathbf{d}_1 = \begin{pmatrix}0\\4\\-3\end{pmatrix}$ (x fixed: 0), $\mathbf{d}_2 = \begin{pmatrix}1\\2\\2\end{pmatrix}$ — x fixed: 0; denominators

$\mathbf{d}_1 \cdot \mathbf{d}_2 = 0\times1 + 4\times2 + (-3)\times2 = 0 + 8 - 6 = 2$ — multiply matching components; add

$|\mathbf{d}_1| = \sqrt{(0^2 + 4^2 + (-3)^2)} = 5, |\mathbf{d}_2| = \sqrt{(1^2 + 2^2 + 2^2)} = 3$ — magnitudes

$\cos\theta = \frac{2}{5\times3}$ — $\cos\theta = |\mathbf{d}_1\cdot\mathbf{d}_2| \div |\mathbf{d}_1||\mathbf{d}_2|$

$\theta = 82.3°$ — inverse cos, 1 d.p.



picture: the acute angle

8 The line l_1 passes through A(1, 0, −2) and B(3, −1, 0). The line l_2 is $x - 4 = \frac{y}{4} = \frac{z+3}{-8}$. Find the acute angle between l_1 and l_2.

$\overrightarrow{AB} = \begin{pmatrix}3\\-1\\0\end{pmatrix} - \begin{pmatrix}1\\0\\-2\end{pmatrix} = \begin{pmatrix}2\\-1\\2\end{pmatrix}$ — direction: AB = b − a

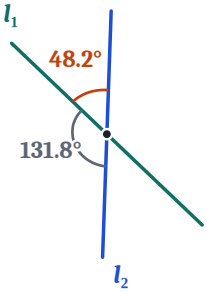
Direction of l_2: $\begin{pmatrix}1\\4\\-8\end{pmatrix}$ — denominators: direction

$\overrightarrow{AB} \cdot \mathbf{d}_2 = 2\times1 + (-1)\times4 + 2\times(-8) = 2 - 4 - 16 = -18$ — multiply matching components; add

$|\overrightarrow{AB}| = \sqrt{(2^2 + (-1)^2 + 2^2)} = 3, |\mathbf{d}_2| = \sqrt{(1^2 + 4^2 + (-8)^2)} = 9$ — magnitudes

$\cos\theta = \frac{|-18|}{3\times9} = \frac{18}{3\times9}$ — modulus: the acute angle

$\theta = 48.2°$ — inverse cos, 1 d.p.



picture: the acute angle

C · Perpendicular lines

Perpendicular ⇔ $\mathbf{d}_1 \cdot \mathbf{d}_2 = 0$: set it equal to 0 and solve.

9 Show that the lines l_1: $\mathbf{r} = \begin{pmatrix}1\\2\\3\end{pmatrix} + \lambda\begin{pmatrix}2\\-1\\3\end{pmatrix}$ and l_2: $\dfrac{x-1}{4} = \dfrac{y+2}{5} = \dfrac{z-1}{-1}$ are perpendicular.

Directions: $\mathbf{d}_1 = \begin{pmatrix}2\\-1\\3\end{pmatrix}$, $\mathbf{d}_2 = \begin{pmatrix}4\\5\\-1\end{pmatrix}$ — read both directions

$\mathbf{d}_1 \cdot \mathbf{d}_2 = 2 \times 4 + (-1) \times 5 + 3 \times (-1)$

multiply matching components

$= 8 - 5 - 3 = 0$ — add

The scalar product is 0, so the lines are perpendicular.

$\mathbf{d}_1 \cdot \mathbf{d}_2 = 0$: perpendicular

10 The lines $\mathbf{r} = \begin{pmatrix}2\\0\\1\end{pmatrix} + \lambda\begin{pmatrix}a\\2\\-1\end{pmatrix}$ and $\mathbf{r} = \begin{pmatrix}-1\\3\\0\end{pmatrix} + \mu\begin{pmatrix}3\\-4\\a\end{pmatrix}$ are perpendicular. Find a.

$\begin{pmatrix}a\\2\\-1\end{pmatrix} \cdot \begin{pmatrix}3\\-4\\a\end{pmatrix} = 3a - 8 - a = 2a - 8$ — multiply matching components

$2a - 8 = 0$ — perpendicular: set equal to 0

$a = 4$ — solve

11 The line l_1: $\dfrac{x-1}{p} = \dfrac{y+3}{2} = \dfrac{z}{p+1}$ is perpendicular to l_2: $\mathbf{r} = \begin{pmatrix}0\\1\\4\end{pmatrix} + \mu\begin{pmatrix}2\\-5\\1\end{pmatrix}$. Find p.

Direction of l_1: $\begin{pmatrix}p\\2\\p+1\end{pmatrix}$ — denominators: direction

$\begin{pmatrix}p\\2\\p+1\end{pmatrix} \cdot \begin{pmatrix}2\\-5\\1\end{pmatrix} = 2p - 10 + p + 1 = 3p - 9$

multiply matching components

$3p - 9 = 0 \Rightarrow p = 3$ — set equal to 0; solve

12 The points $A(1, k, 0)$ and $B(3, 1, 2)$ lie on the line l_1, which is perpendicular to l_2: $\mathbf{r} = \begin{pmatrix}5\\0\\1\end{pmatrix} + \mu\begin{pmatrix}3\\2\\-1\end{pmatrix}$. Find k.

$\overrightarrow{AB} = \begin{pmatrix}3\\1\\2\end{pmatrix} - \begin{pmatrix}1\\k\\0\end{pmatrix} = \begin{pmatrix}2\\1-k\\2\end{pmatrix}$ — direction: AB = b − a

$\begin{pmatrix}2\\1-k\\2\end{pmatrix} \cdot \begin{pmatrix}3\\2\\-1\end{pmatrix} = 6 + 2(1-k) - 2 = 6 - 2k$ — multiply; expand

$6 - 2k = 0 \Rightarrow k = 3$ — set equal to 0; solve

D · Axes, skew lines and reasoning Explain in words as well as numbers.

13 The line l_1 has direction $\mathbf{i} + 4\mathbf{j} + \mathbf{k}$. Beth writes the direction of l_2 as $2\mathbf{i} - \mathbf{j} + 3\mathbf{k}$; Amir writes it as $-4\mathbf{i} + 2\mathbf{j} - 6\mathbf{k}$.
(a) Use Beth's vector to find the acute angle between l_1 and l_2.
(b) Show that Amir's vector gives the same angle.

(a) $\mathbf{b}\cdot\mathbf{d}_1 = 2\times1 + (-1)\times4 + 3\times1 = 2 - 4 + 3 = 1$

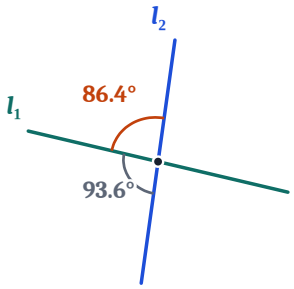
multiply matching components; add

$|\mathbf{b}| = \sqrt{(2^2 + (-1)^2 + 3^2)} = \sqrt{14}$, $|\mathbf{d}_1| = \sqrt{(1^2 + 4^2 + 1^2)} = \sqrt{18}$

magnitudes

$\cos\theta = \dfrac{1}{\sqrt{14}\times\sqrt{18}}$ — $\cos\theta = |d_1\cdot d_2| \div |d_1||d_2|$

$\theta = 86.4°$ — inverse cos, 1 d.p.



picture: the acute angle

(b) $\begin{pmatrix}-4\\2\\-6\end{pmatrix} = -2\begin{pmatrix}2\\-1\\3\end{pmatrix}$: the scalar product is -2 times as big, the magnitude 2 times as big.

Amir: a multiple of Beth

$\cos\theta = \dfrac{|-2|}{\sqrt{56}\times\sqrt{18}} = \dfrac{2}{2\sqrt{14}\times\sqrt{18}} = \dfrac{1}{\sqrt{14}\times\sqrt{18}}$

modulus: the same cos θ

The same $\cos\theta$, so the same angle, 86.4°.

same line, same angle

14 The lines l_1: $\mathbf{r} = \begin{pmatrix}1\\0\\2\end{pmatrix} + \lambda\begin{pmatrix}6\\2\\-3\end{pmatrix}$ and l_2: $\mathbf{r} = \begin{pmatrix}0\\1\\-1\end{pmatrix} + \mu\begin{pmatrix}0\\4\\3\end{pmatrix}$ are skew. Hana says skew lines have no angle between them. Explain why she is wrong, and find the acute angle.

The angle between two lines depends only on their directions. Slide l_2 (without turning it) until it meets l_1: the angle there is the angle between the lines.

directions only

$\mathbf{d}_1\cdot\mathbf{d}_2 = 6\times0 + 2\times4 + (-3)\times3 = 0 + 8 - 9 = -1$

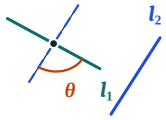
multiply matching components; add

$|\mathbf{d}_1| = \sqrt{(6^2 + 2^2 + (-3)^2)} = 7$, $|\mathbf{d}_2| = \sqrt{(0^2 + 4^2 + 3^2)} = 5$

magnitudes

$\cos\theta = \dfrac{|-1|}{7\times5} = \dfrac{1}{7\times5}$ — modulus: the acute angle

$\theta = 88.4°$ — inverse cos, 1 d.p.



sketch: slide l_2 to meet l_1

15 Find the acute angle between the line $\dfrac{x+2}{2} = \dfrac{1-y}{3} = \dfrac{z-4}{6}$ and the y-axis.

$\dfrac{1-y}{3} = \dfrac{y-1}{-3}$ — top and bottom × (−1)

Direction $\begin{pmatrix}2\\-3\\6\end{pmatrix}$ — denominators: direction

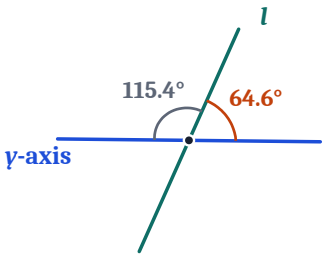
y-axis: direction $\mathbf{j}$, $\mathbf{d}\cdot\mathbf{j} = -3$ (the y-component)

y-axis: j; d·j = y-component

$|\mathbf{d}| = \sqrt{(2^2 + (-3)^2 + 6^2)} = 7$ — magnitude

$\cos\theta = \dfrac{|-3|}{7} = \dfrac{3}{7}$ — modulus; $|\mathbf{j}| = 1$

$\theta = 64.6°$ — inverse cos, 1 d.p.



picture: the angle with the y-axis

16 The line l has direction $a\mathbf{i} + 3\mathbf{j} - 4\mathbf{k}$ and is perpendicular to the x-axis.
(a) Find a.
(b) Find the acute angle between l and the y-axis.

(a) $\begin{pmatrix}a\\3\\-4\end{pmatrix}\cdot\begin{pmatrix}1\\0\\0\end{pmatrix} = a$ — d·i: the x-component

$a = 0$ — perpendicular: set equal to 0

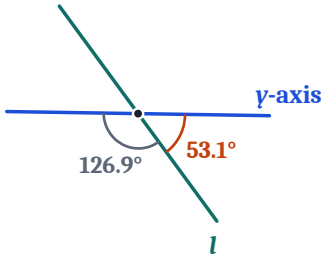
(b) y-axis: direction $\mathbf{j}$, $\mathbf{d}\cdot\mathbf{j} = 3$ (the y-component)

y-axis: j; d·j = y-component

$|\mathbf{d}| = \sqrt{(0^2 + 3^2 + (-4)^2)} = 5$ — magnitude

$\cos\theta = \dfrac{3}{5}$ — $|\mathbf{j}| = 1$

$\theta = 53.1°$ — inverse cos, 1 d.p.



picture: the angle with the y-axis

E · Exam-style question Show your method: marks are for working.

17 The lines l_1 and l_2 have equations

l_1: $\mathbf{r} = \begin{pmatrix} 2 \\ -1 \\ 3 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 2 \\ -2 \end{pmatrix}$ and l_2: $\dfrac{x-5}{4} = \dfrac{y+1}{-1} = \dfrac{z-2}{8}$.

(a) Find the acute angle between l_1 and l_2, giving your answer to the nearest $0.1°$. [3 marks]

(b) The line l_3 has direction $4\mathbf{i} + p\mathbf{j} + \mathbf{k}$ and is perpendicular to l_1. Find p. [2 marks]

(c) A student uses $2\mathbf{i} - \mathbf{j} + 3\mathbf{k}$ to find the angle in part (a). Explain why this is wrong. [1 mark]

(d) Find the acute angle between l_2 and the z-axis. [2 marks]

(a) $\mathbf{d}_1 \cdot \mathbf{d}_2 = 1 \times 4 + 2 \times (-1) + (-2) \times 8 = -14$ **M1** — scalar product of directions

$\cos\theta = \dfrac{|-14|}{3 \times 9} = \dfrac{14}{27}$ **M1** — modulus; $|\mathbf{d}_1| = 3$, $|\mathbf{d}_2| = 9$

$\theta = 58.8°$ **A1** — inverse cos, 1 d.p.

(b) $1 \times 4 + 2 \times p + (-2) \times 1 = 2p + 2 = 0$ **M1** — set $\mathbf{d}_1 \cdot \mathbf{d}_3 = 0$

$p = -1$ **A1** — solve

(c) It is the position vector of a point on l_1: it fixes where the line is, not which way it points. The angle depends only on the direction vectors. **E1** — position ≠ direction

(d) $\mathbf{d}_2 \cdot \mathbf{k} = 8$, $|\mathbf{d}_2| = 9$ **M1** — d·k: the z-component

$\cos\theta = \dfrac{8}{9}$, $\theta = 27.3°$ **A1** — inverse cos, 1 d.p.