Shortest Distances

Specification: AQA 7366/7367: F1, F5 · Edexcel 9FM0: Core Pure 6.5

INDEPENDENT **You do**

A · Point to a line M on the line, PM·d = 0, then |PM|. Exact answers.

1 Find the foot of the perpendicular from P(4, 2, 1) to l: $\mathbf{r} = \begin{pmatrix}1\\1\\0\end{pmatrix} + \lambda\begin{pmatrix}1\\0\\1\end{pmatrix}$, and the shortest distance from P to l.

M on l: $\overrightarrow{PM} = \begin{pmatrix}1+\lambda\\1\\\lambda\end{pmatrix} - \begin{pmatrix}4\\2\\1\end{pmatrix} = \begin{pmatrix}-3+\lambda\\-1\\-1+\lambda\end{pmatrix}$

M on the line: PM = m − p

$\overrightarrow{PM}\cdot\mathbf{d} = (-3+\lambda) + (-1+\lambda) = 2\lambda - 4 = 0 \Rightarrow \lambda = 2$

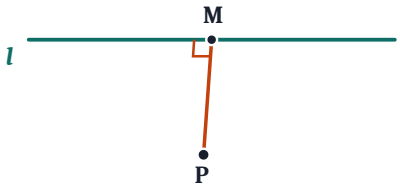
PM ⊥ line: PM·d = 0

$\lambda = 2$: $\overrightarrow{OM} = \begin{pmatrix}1\\1\\0\end{pmatrix} + 2\begin{pmatrix}1\\0\\1\end{pmatrix} = \begin{pmatrix}3\\1\\2\end{pmatrix}$, M(3, 1, 2)

substitute λ: the foot M

$\overrightarrow{PM} = \begin{pmatrix}3\\1\\2\end{pmatrix} - \begin{pmatrix}4\\2\\1\end{pmatrix} = \begin{pmatrix}-1\\-1\\1\end{pmatrix}$, $|\overrightarrow{PM}| = \sqrt{(1+1+1)} = \sqrt{3} = 1.73$ (3 s.f.)

distance: |PM|



picture: PM ⊥ the line

2 Find the foot of the perpendicular from P(3, −2, 7) to l: $\mathbf{r} = \begin{pmatrix}2\\-1\\3\end{pmatrix} + \lambda\begin{pmatrix}1\\2\\-2\end{pmatrix}$, and the shortest distance from P to l.

M on l: $\overrightarrow{PM} = \begin{pmatrix}2+\lambda\\-1+2\lambda\\3-2\lambda\end{pmatrix} - \begin{pmatrix}3\\-2\\7\end{pmatrix} = \begin{pmatrix}-1+\lambda\\1+2\lambda\\-4-2\lambda\end{pmatrix}$

M on the line: PM = m − p

$\overrightarrow{PM}\cdot\mathbf{d} = (-1+\lambda) + 2(1+2\lambda) - 2(-4-2\lambda) = 9\lambda + 9 = 0$
$\Rightarrow \lambda = -1$

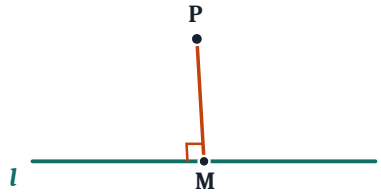
PM ⊥ line: PM·d = 0

$\lambda = -1$: $\overrightarrow{OM} = \begin{pmatrix}2\\-1\\3\end{pmatrix} - \begin{pmatrix}1\\2\\-2\end{pmatrix} = \begin{pmatrix}1\\-3\\5\end{pmatrix}$, M(1, −3, 5)

substitute λ: the foot M

$\overrightarrow{PM} = \begin{pmatrix}1\\-3\\5\end{pmatrix} - \begin{pmatrix}3\\-2\\7\end{pmatrix} = \begin{pmatrix}-2\\-1\\-2\end{pmatrix}$, $|\overrightarrow{PM}| = \sqrt{(4+1+4)} = 3$

distance: |PM|



picture: PM ⊥ the line

3 Find the foot of the perpendicular from P(4, 6, 0) to l: $\mathbf{r} = \begin{pmatrix}0\\4\\-2\end{pmatrix} + \lambda\begin{pmatrix}3\\0\\-1\end{pmatrix}$, and the shortest distance from P to l.

M on l: $\overrightarrow{PM} = \begin{pmatrix}3\lambda\\4\\-2-\lambda\end{pmatrix} - \begin{pmatrix}4\\6\\0\end{pmatrix} = \begin{pmatrix}-4+3\lambda\\-2\\-2-\lambda\end{pmatrix}$

M on the line: PM = m − p

$\overrightarrow{PM}\cdot\mathbf{d} = 3(-4+3\lambda) - (-2-\lambda) = 10\lambda - 10 = 0 \Rightarrow \lambda = 1$

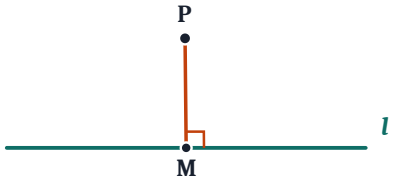
PM ⊥ line: PM·d = 0

$\lambda = 1$: $\overrightarrow{OM} = \begin{pmatrix}0\\4\\-2\end{pmatrix} + \begin{pmatrix}3\\0\\-1\end{pmatrix} = \begin{pmatrix}3\\4\\-3\end{pmatrix}$, M(3, 4, −3)

substitute λ: the foot M

$\overrightarrow{PM} = \begin{pmatrix}3\\4\\-3\end{pmatrix} - \begin{pmatrix}4\\6\\0\end{pmatrix} = \begin{pmatrix}-1\\-2\\-3\end{pmatrix}$, $|\overrightarrow{PM}| = \sqrt{(1+4+9)}$
$= \sqrt{14} = 3.74$ (3 s.f.)

distance: |PM|



picture: PM ⊥ the line

4 Find the foot of the perpendicular from P(−2, −8, 0) to l: $\mathbf{r} = (\mathbf{i} - 2\mathbf{j}) + \lambda(2\mathbf{i} + 2\mathbf{j} + \mathbf{k})$, and the shortest distance from P to l.

M on l: $\overrightarrow{PM} = \begin{pmatrix}1+2\lambda\\-2+2\lambda\\\lambda\end{pmatrix} - \begin{pmatrix}-2\\-8\\0\end{pmatrix} = \begin{pmatrix}3+2\lambda\\6+2\lambda\\\lambda\end{pmatrix}$

M on the line: PM = m − p

$\overrightarrow{PM}\cdot\mathbf{d} = 2(3+2\lambda) + 2(6+2\lambda) + \lambda = 9\lambda + 18 = 0 \Rightarrow \lambda = -2$

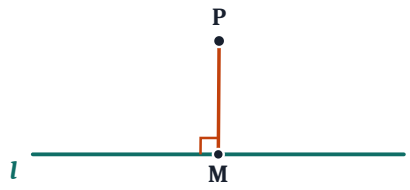
PM ⊥ line: PM·d = 0

$\lambda = -2$: $\overrightarrow{OM} = \begin{pmatrix}1\\-2\\0\end{pmatrix} - 2\begin{pmatrix}2\\2\\1\end{pmatrix} = \begin{pmatrix}-3\\-6\\-2\end{pmatrix}$, M(−3, −6, −2)

substitute λ: the foot M

$\overrightarrow{PM} = \begin{pmatrix}-3\\-6\\-2\end{pmatrix} - \begin{pmatrix}-2\\-8\\0\end{pmatrix} = \begin{pmatrix}-1\\2\\-2\end{pmatrix}$, $|\overrightarrow{PM}| = \sqrt{(1+4+4)} = 3$

distance: |PM|



picture: PM ⊥ the line

B · Other forms, the origin and reflections Find a point and a direction first.

5 The line l passes through A$(2, 1, -1)$ and B$(4, 2, 1)$. Find the shortest distance from P$(1, 0, -4)$ to l.

$\overrightarrow{AB} = \begin{pmatrix}4\\2\\1\end{pmatrix} - \begin{pmatrix}2\\1\\-1\end{pmatrix} = \begin{pmatrix}2\\1\\2\end{pmatrix}$ direction: AB = b − a

M on l: $\overrightarrow{PM} = \begin{pmatrix}2+2\lambda\\1+\lambda\\-1+2\lambda\end{pmatrix} - \begin{pmatrix}1\\0\\-4\end{pmatrix} = \begin{pmatrix}1+2\lambda\\1+\lambda\\3+2\lambda\end{pmatrix}$

M on the line: PM = m − p

$\overrightarrow{PM}\cdot\mathbf{d} = 2(1+2\lambda) + (1+\lambda) + 2(3+2\lambda) = 9\lambda + 9 = 0$

$\Rightarrow \lambda = -1$

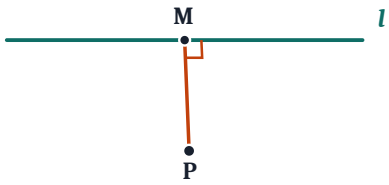
$\lambda = -1$: $\overrightarrow{OM} = \begin{pmatrix}2\\1\\-1\end{pmatrix} - \begin{pmatrix}2\\1\\2\end{pmatrix} = \begin{pmatrix}0\\0\\-3\end{pmatrix}$, M$(0, 0, -3)$

PM·d = 0; substitute

$\overrightarrow{PM} = \begin{pmatrix}0\\0\\-3\end{pmatrix} - \begin{pmatrix}1\\0\\-4\end{pmatrix} = \begin{pmatrix}-1\\0\\1\end{pmatrix}$, $|\overrightarrow{PM}| = \sqrt{(1+0+1)}$

$= \sqrt{2} = 1.41$ (3 s.f.)

distance: |PM|



picture: PM ⊥ the line

6 Find the shortest distance from P$(2, 3, 7)$ to the line l: $\dfrac{x+1}{2} = \dfrac{3-y}{2} = z - 4$.

$\dfrac{3-y}{2} = \dfrac{y-3}{-2}$; l: point $(-1, 3, 4)$, direction $\begin{pmatrix}2\\-2\\1\end{pmatrix}$

rewrite; point and direction

M on l: $\overrightarrow{PM} = \begin{pmatrix}-1+2\lambda\\3-2\lambda\\4+\lambda\end{pmatrix} - \begin{pmatrix}2\\3\\7\end{pmatrix} = \begin{pmatrix}-3+2\lambda\\-2\lambda\\-3+\lambda\end{pmatrix}$

M on the line: PM = m − p

$\overrightarrow{PM}\cdot\mathbf{d} = 2(-3+2\lambda) - 2(-2\lambda) + (-3+\lambda) = 9\lambda - 9 = 0$

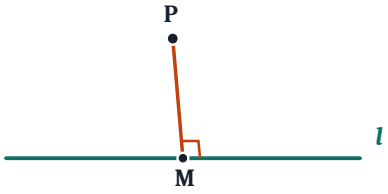
$\Rightarrow \lambda = 1$

$\lambda = 1$: $\overrightarrow{OM} = \begin{pmatrix}-1\\3\\4\end{pmatrix} + \begin{pmatrix}2\\-2\\1\end{pmatrix} = \begin{pmatrix}1\\1\\5\end{pmatrix}$, M$(1, 1, 5)$

PM·d = 0; substitute

$\overrightarrow{PM} = \begin{pmatrix}1\\1\\5\end{pmatrix} - \begin{pmatrix}2\\3\\7\end{pmatrix} = \begin{pmatrix}-1\\-2\\-2\end{pmatrix}$, $|\overrightarrow{PM}| = \sqrt{(1+4+4)} = 3$

distance: |PM|



picture: PM ⊥ the line

7 Find the point of the line $\mathbf{r} = \begin{pmatrix}0\\-5\\-4\end{pmatrix} + \lambda\begin{pmatrix}1\\2\\2\end{pmatrix}$ that is closest to the origin O, and its distance from O.

M on l: $\overrightarrow{OM} = \begin{pmatrix}\lambda\\-5+2\lambda\\-4+2\lambda\end{pmatrix} - \begin{pmatrix}0\\0\\0\end{pmatrix} = \begin{pmatrix}\lambda\\-5+2\lambda\\-4+2\lambda\end{pmatrix}$

M on the line: OM = m

$\overrightarrow{OM}\cdot\mathbf{d} = \lambda + 2(-5+2\lambda) + 2(-4+2\lambda) = 9\lambda - 18 = 0 \Rightarrow \lambda = 2$

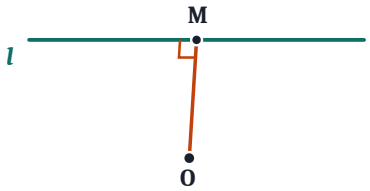
P is O here: OM·d = 0

$\lambda = 2$: $\overrightarrow{OM} = \begin{pmatrix}0\\-5\\-4\end{pmatrix} + 2\begin{pmatrix}1\\2\\2\end{pmatrix} = \begin{pmatrix}2\\-1\\0\end{pmatrix}$, M$(2, -1, 0)$

substitute λ: the foot M

$\overrightarrow{OM} = \begin{pmatrix}2\\-1\\0\end{pmatrix} - \begin{pmatrix}0\\0\\0\end{pmatrix} = \begin{pmatrix}2\\-1\\0\end{pmatrix}$, $|\overrightarrow{OM}| = \sqrt{(4+1+0)} = \sqrt{5} = 2.24$ (3 s.f.)

distance: |OM|



picture: OM ⊥ the line

8 Find the image of P$(3, 1, 5)$ after reflection in the line l: $\mathbf{r} = \begin{pmatrix}1\\0\\3\end{pmatrix} + \lambda\begin{pmatrix}1\\-1\\1\end{pmatrix}$.

M on l: $\overrightarrow{PM} = \begin{pmatrix}1+\lambda\\-\lambda\\3+\lambda\end{pmatrix} - \begin{pmatrix}3\\1\\5\end{pmatrix} = \begin{pmatrix}-2+\lambda\\-1-\lambda\\-2+\lambda\end{pmatrix}$

M on the line: PM = m − p

$\overrightarrow{PM}\cdot\mathbf{d} = (-2+\lambda) - (-1-\lambda) + (-2+\lambda) = 3\lambda - 3 = 0 \Rightarrow \lambda = 1$

PM ⊥ line: PM·d = 0

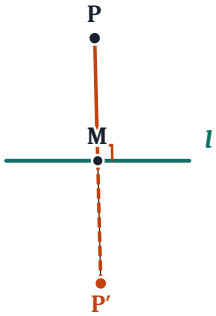
$\lambda = 1$: $\overrightarrow{OM} = \begin{pmatrix}1\\0\\3\end{pmatrix} + \begin{pmatrix}1\\-1\\1\end{pmatrix} = \begin{pmatrix}2\\-1\\4\end{pmatrix}$, M$(2, -1, 4)$

substitute λ: the foot M

M is the midpoint of PP′:

$\mathbf{p}' = 2\mathbf{m} - \mathbf{p} = 2\begin{pmatrix}2\\-1\\4\end{pmatrix} - \begin{pmatrix}3\\1\\5\end{pmatrix} = \begin{pmatrix}1\\-3\\3\end{pmatrix}$, P′$(1, -3, 3)$

M midpoint: p′ = 2m − p



picture: M is the midpoint of PP′

C · Parallel lines Any point of one line; its distance to the other line.

9 Show that l_1: $\mathbf{r} = \begin{pmatrix}1\\1\\1\end{pmatrix} + \lambda\begin{pmatrix}1\\2\\-1\end{pmatrix}$ and l_2: $\mathbf{r} = \begin{pmatrix}3\\3\\1\end{pmatrix} + \mu\begin{pmatrix}2\\4\\-2\end{pmatrix}$ are parallel, and find the distance between them.

$\begin{pmatrix}2\\4\\-2\end{pmatrix} = 2\begin{pmatrix}1\\2\\-1\end{pmatrix}$: parallel. The distance between the lines is the distance from B$(3,3,1)$ on l_2 to l_1.

parallel: distance from B to l_1

M on l_1: $\overrightarrow{BM} = \begin{pmatrix}1+\lambda\\1+2\lambda\\1-\lambda\end{pmatrix} - \begin{pmatrix}3\\3\\1\end{pmatrix} = \begin{pmatrix}-2+\lambda\\-2+2\lambda\\-\lambda\end{pmatrix}$

M on the line: BM = m − b

$\overrightarrow{BM}\cdot\mathbf{d}_1 = (-2+\lambda) + 2(-2+2\lambda) - (-\lambda) = 6\lambda - 6 = 0 \Rightarrow \lambda = 1$

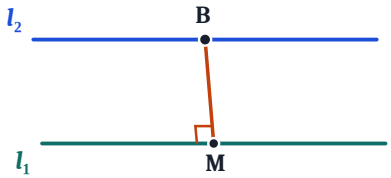
BM ⊥ line: BM·d = 0

$\lambda = 1$: $\overrightarrow{OM} = \begin{pmatrix}1\\1\\1\end{pmatrix} + \begin{pmatrix}1\\2\\-1\end{pmatrix} = \begin{pmatrix}2\\3\\0\end{pmatrix}$, M$(2,3,0)$

substitute λ: the foot M

$\overrightarrow{BM} = \begin{pmatrix}2\\3\\0\end{pmatrix} - \begin{pmatrix}3\\3\\1\end{pmatrix} = \begin{pmatrix}-1\\0\\-1\end{pmatrix}$, $|\overrightarrow{BM}| = \sqrt{(1+0+1)} = \sqrt{2} = 1.41$ (3 s.f.)

distance: |BM|



picture: the gap between the lines

10 Show that l_1: $\mathbf{r} = (3\mathbf{i} - \mathbf{k}) + \lambda(2\mathbf{i} - 2\mathbf{j} + \mathbf{k})$ and l_2: $\mathbf{r} = (3\mathbf{i} + 3\mathbf{j} - 4\mathbf{k}) + \mu(-2\mathbf{i} + 2\mathbf{j} - \mathbf{k})$ are parallel, and find the distance between them.

$\begin{pmatrix}-2\\2\\-1\end{pmatrix} = -\begin{pmatrix}2\\-2\\1\end{pmatrix}$: parallel. The distance between the lines is the distance from B$(3,3,-4)$ on l_2 to l_1.

parallel: distance from B to l_1

M on l_1: $\overrightarrow{BM} = \begin{pmatrix}3+2\lambda\\-2\lambda\\-1+\lambda\end{pmatrix} - \begin{pmatrix}3\\3\\-4\end{pmatrix} = \begin{pmatrix}2\lambda\\-3-2\lambda\\3+\lambda\end{pmatrix}$

M on the line: BM = m − b

$\overrightarrow{BM}\cdot\mathbf{d}_1 = 2(2\lambda) - 2(-3-2\lambda) + (3+\lambda) = 9\lambda + 9 = 0 \Rightarrow \lambda = -1$

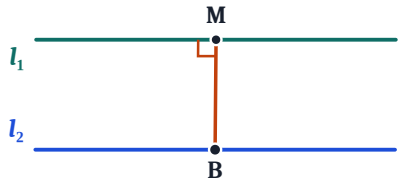
BM ⊥ line: BM·d = 0

$\lambda = -1$: $\overrightarrow{OM} = \begin{pmatrix}3\\0\\-1\end{pmatrix} - \begin{pmatrix}2\\-2\\1\end{pmatrix} = \begin{pmatrix}1\\2\\-2\end{pmatrix}$, M$(1,2,-2)$

substitute λ: the foot M

$\overrightarrow{BM} = \begin{pmatrix}1\\2\\-2\end{pmatrix} - \begin{pmatrix}3\\3\\-4\end{pmatrix} = \begin{pmatrix}-2\\-1\\2\end{pmatrix}$, $|\overrightarrow{BM}| = \sqrt{(4+1+4)} = 3$

distance: |BM|



picture: the gap between the lines

11 Show that l_1: $\frac{x-1}{2} = y+1 = \frac{z-2}{-2}$ and l_2: $\mathbf{r} = \begin{pmatrix}8\\-1\\0\end{pmatrix} + \mu\begin{pmatrix}4\\2\\-4\end{pmatrix}$ are parallel, and find the distance between them.

l_1: point $(1, -1, 2)$, direction $\begin{pmatrix}2\\1\\-2\end{pmatrix}$ — Cartesian: point, direction

$\begin{pmatrix}4\\2\\-4\end{pmatrix} = 2\begin{pmatrix}2\\1\\-2\end{pmatrix}$: parallel. The distance between the lines is the distance from B$(8, -1, 0)$ on l_2 to l_1.

parallel: distance from B to l_1

M on l_1: $\overrightarrow{BM} = \begin{pmatrix}1+2\lambda\\-1+\lambda\\2-2\lambda\end{pmatrix} - \begin{pmatrix}8\\-1\\0\end{pmatrix} = \begin{pmatrix}-7+2\lambda\\\lambda\\2-2\lambda\end{pmatrix}$

M on the line: BM = m − b

$\overrightarrow{BM}\cdot\mathbf{d}_1 = 2(-7+2\lambda) + \lambda - 2(2-2\lambda) = 9\lambda - 18 = 0 \Rightarrow \lambda = 2$

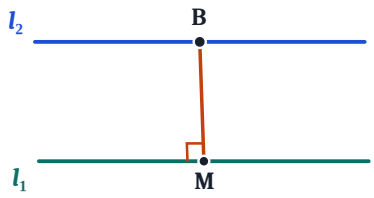
BM ⊥ line: BM·d = 0

$\lambda = 2$: $\overrightarrow{OM} = \begin{pmatrix}1\\-1\\2\end{pmatrix} + 2\begin{pmatrix}2\\1\\-2\end{pmatrix} = \begin{pmatrix}5\\1\\-2\end{pmatrix}$, M$(5, 1, -2)$

substitute λ: the foot M

$\overrightarrow{BM} = \begin{pmatrix}5\\1\\-2\end{pmatrix} - \begin{pmatrix}8\\-1\\0\end{pmatrix} = \begin{pmatrix}-3\\2\\-2\end{pmatrix}$, $|\overrightarrow{BM}| = \sqrt{(9+4+4)}$ $= \sqrt{17} = 4.12$ (3 s.f.)

distance: |BM|



picture: the gap between the lines

D · Skew lines P on l_1 and Q on l_2 both move: PQ·d_1 = 0 and PQ·d_2 = 0.

12 Find the shortest distance between l_1: $\mathbf{r} = \begin{pmatrix}2\\0\\1\end{pmatrix} + \lambda\begin{pmatrix}1\\-1\\2\end{pmatrix}$ and l_2: $\mathbf{r} = \begin{pmatrix}3\\1\\4\end{pmatrix} + \mu\begin{pmatrix}1\\1\\0\end{pmatrix}$.

$\overrightarrow{PQ} = \begin{pmatrix}3+\mu\\1+\mu\\4\end{pmatrix} - \begin{pmatrix}2+\lambda\\-\lambda\\1+2\lambda\end{pmatrix} = \begin{pmatrix}1-\lambda+\mu\\1+\lambda+\mu\\3-2\lambda\end{pmatrix}$ — P on l_1, Q on l_2: PQ = q − p

$\overrightarrow{PQ}\cdot\mathbf{d}_1 = (1-\lambda+\mu) - (1+\lambda+\mu) + 2(3-2\lambda)$
$= -6\lambda + 6 = 0 \Rightarrow \lambda = 1$ (1)

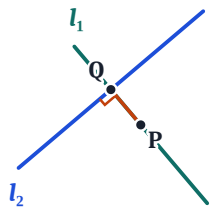
PQ ⊥ l_1: PQ·d_1 = 0

$\overrightarrow{PQ}\cdot\mathbf{d}_2 = (1-\lambda+\mu) + (1+\lambda+\mu)$
$= 2\mu + 2 = 0 \Rightarrow \mu = -1$ (2)

PQ ⊥ l_2: PQ·d_2 = 0

$\lambda = 1, \mu = -1$: $\overrightarrow{PQ} = \begin{pmatrix}-1\\1\\1\end{pmatrix}$, $|\overrightarrow{PQ}| = \sqrt{(1+1+1)} = \sqrt{3} = 1.73$ (3 s.f.)

shortest distance: |PQ|



picture: PQ ⊥ both lines

13 Find the shortest distance between l_1: $\mathbf{r} = \begin{pmatrix}0\\1\\-2\end{pmatrix} + \lambda\begin{pmatrix}2\\1\\0\end{pmatrix}$ and l_2: $\mathbf{r} = \begin{pmatrix}2\\1\\-6\end{pmatrix} + \mu\begin{pmatrix}1\\-1\\1\end{pmatrix}$.

$\overrightarrow{PQ} = \begin{pmatrix}2+\mu\\1-\mu\\-6+\mu\end{pmatrix} - \begin{pmatrix}2\lambda\\1+\lambda\\-2\end{pmatrix} = \begin{pmatrix}2-2\lambda+\mu\\-\lambda-\mu\\-4+\mu\end{pmatrix}$ — P on l_1, Q on l_2: PQ = q − p

$\overrightarrow{PQ}\cdot\mathbf{d}_1 = 2(2-2\lambda+\mu) + (-\lambda-\mu)$
$= -5\lambda + \mu + 4 = 0 \Rightarrow 5\lambda - \mu = 4$ (1)

PQ ⊥ l_1: PQ·d_1 = 0

$\overrightarrow{PQ}\cdot\mathbf{d}_2 = (2-2\lambda+\mu) - (-\lambda-\mu) + (-4+\mu)$
$= -\lambda + 3\mu - 2 = 0 \Rightarrow \lambda - 3\mu = -2$ (2)

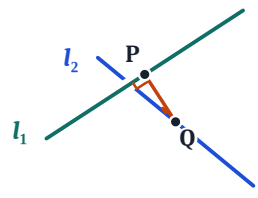
PQ ⊥ l_2: PQ·d_2 = 0

3 × (1) − (2): $14\lambda = 14 \Rightarrow \lambda = 1$; (1): $5(1) - \mu = 4 \Rightarrow \mu = 1$

subtract: μ cancels; substitute

$\lambda = 1, \mu = 1$: $\overrightarrow{PQ} = \begin{pmatrix}1\\-2\\-3\end{pmatrix}$, $|\overrightarrow{PQ}| = \sqrt{(1+4+9)} = \sqrt{14} = 3.74$ (3 s.f.)

shortest distance: |PQ|



picture: PQ ⊥ both lines

14 Find the points P on l_1 and Q on l_2 that are closest together, and the shortest distance, for $l_1: \mathbf{r} = \begin{pmatrix}3\\-1\\0\end{pmatrix} + \lambda\begin{pmatrix}1\\2\\-2\end{pmatrix}$ and $l_2: \mathbf{r} = \begin{pmatrix}4\\-9\\-2\end{pmatrix} + \mu\begin{pmatrix}2\\1\\1\end{pmatrix}$.

$\overrightarrow{PQ} = \begin{pmatrix}4+2\mu\\-9+\mu\\-2+\mu\end{pmatrix} - \begin{pmatrix}3+\lambda\\-1+2\lambda\\-2\lambda\end{pmatrix} = \begin{pmatrix}1-\lambda+2\mu\\-8-2\lambda+\mu\\-2+2\lambda+\mu\end{pmatrix}$

P on l_1, Q on l_2: PQ = q − p

$\overrightarrow{PQ}\cdot\mathbf{d}_1 = (1-\lambda+2\mu) + 2(-8-2\lambda+\mu) - 2(-2+2\lambda+\mu)$
$= -9\lambda + 2\mu - 11 = 0 \Rightarrow 9\lambda - 2\mu = -11$ (1)

PQ ⊥ l_1: PQ·d_1 = 0

$\overrightarrow{PQ}\cdot\mathbf{d}_2 = 2(1-\lambda+2\mu) + (-8-2\lambda+\mu) + (-2+2\lambda+\mu)$
$= -2\lambda + 6\mu - 8 = 0 \Rightarrow \lambda - 3\mu = -4$ (2)

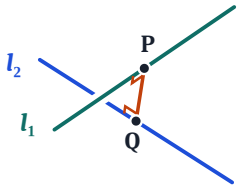
PQ ⊥ l_2: PQ·d_2 = 0

$3\times$(1) − 2 × (2): $25\lambda = -25 \Rightarrow \lambda = -1$;
(1): $9(-1) - 2\mu = -11 \Rightarrow \mu = 1 \Rightarrow$ P(2, −3, 2), Q(6, −8, −1)

subtract: μ cancels; substitute; P and Q

$\overrightarrow{PQ} = \begin{pmatrix}4\\-5\\-3\end{pmatrix}$, $|\overrightarrow{PQ}| = \sqrt{(16+25+9)} = \sqrt{50} = 5\sqrt{2} = 7.07$ (3 s.f.)

shortest distance: |PQ|



picture: PQ ⊥ both lines

15 Find the shortest distance between $l_1: \mathbf{r} = \begin{pmatrix}1\\2\\-1\end{pmatrix} + \lambda\begin{pmatrix}1\\0\\2\end{pmatrix}$ and $l_2: \mathbf{r} = \begin{pmatrix}5\\3\\2\end{pmatrix} + \mu\begin{pmatrix}2\\1\\-1\end{pmatrix}$. What does your answer tell you?

$\overrightarrow{PQ} = \begin{pmatrix}5+2\mu\\3+\mu\\2-\mu\end{pmatrix} - \begin{pmatrix}1+\lambda\\2\\-1+2\lambda\end{pmatrix} = \begin{pmatrix}4-\lambda+2\mu\\1+\mu\\3-2\lambda-\mu\end{pmatrix}$

P on l_1, Q on l_2: PQ = q − p

$\overrightarrow{PQ}\cdot\mathbf{d}_1 = (4-\lambda+2\mu) + 2(3-2\lambda-\mu)$
$= -5\lambda + 10 = 0 \Rightarrow \lambda = 2$ (1)

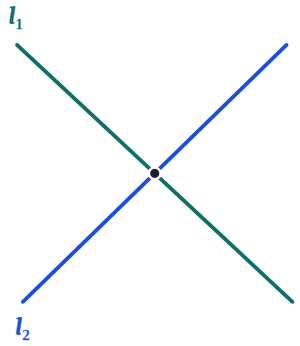
PQ ⊥ l_1: PQ·d_1 = 0

$\overrightarrow{PQ}\cdot\mathbf{d}_2 = 2(4-\lambda+2\mu) + (1+\mu) - (3-2\lambda-\mu)$
$= 6\mu + 6 = 0 \Rightarrow \mu = -1$ (2)

PQ ⊥ l_2: PQ·d_2 = 0

$\lambda = 2, \mu = -1$: $\overrightarrow{PQ} = \begin{pmatrix}0\\0\\0\end{pmatrix}$: P = Q, so the lines meet at (3, 2, 3). The shortest distance is 0.

PQ = 0: the lines meet



picture: the lines meet, distance 0

E · Exam-style question Show your method: marks are for working.

16 The lines l_1 and l_2 have equations $l_1: \mathbf{r} = \begin{pmatrix}1\\1\\-2\end{pmatrix} + \lambda\begin{pmatrix}1\\-1\\2\end{pmatrix}$ and $l_2: \mathbf{r} = \begin{pmatrix}-4\\2\\3\end{pmatrix} + \mu\begin{pmatrix}2\\1\\0\end{pmatrix}$.

(a) Show that l_1 and l_2 are not parallel. [1 mark]

(b) The point P lies on l_1 and the point Q lies on l_2, and PQ is perpendicular to both lines. Find the coordinates of P and Q. [5 marks]

(c) Hence find the exact shortest distance between l_1 and l_2. [2 marks]

(a) $\begin{pmatrix}2\\1\\0\end{pmatrix} = k\begin{pmatrix}1\\-1\\2\end{pmatrix}$ needs $k = 2$ (x) and $k = 0$ (z): not parallel **E1**

directions not multiples

(b) $\overrightarrow{PQ} = \begin{pmatrix}-4+2\mu\\2+\mu\\3\end{pmatrix} - \begin{pmatrix}1+\lambda\\1-\lambda\\-2+2\lambda\end{pmatrix} = \begin{pmatrix}-5-\lambda+2\mu\\1+\lambda+\mu\\5-2\lambda\end{pmatrix}$ **M1**

P on l_1, Q on l_2: PQ = q − p

$\overrightarrow{PQ}\cdot\mathbf{d}_1 = (-5-\lambda+2\mu) - (1+\lambda+\mu) + 2(5-2\lambda)$
$= -6\lambda + \mu + 4 = 0 \Rightarrow 6\lambda - \mu = 4$ (1) **M1**

PQ ⊥ l_1: PQ·d_1 = 0

$\overrightarrow{PQ}\cdot\mathbf{d}_2 = 2(-5-\lambda+2\mu) + (1+\lambda+\mu)$
$= -\lambda + 5\mu - 9 = 0 \Rightarrow \lambda - 5\mu = -9$ (2) **A1**

PQ ⊥ l_2: PQ·d_2 = 0

$5\times$(1) − (2): $29\lambda = 29 \Rightarrow \lambda = 1$; (1): $6(1) - \mu = 4 \Rightarrow \mu = 2$
P(2, 0, 0), Q(0, 4, 3) **A1 A1**

subtract: μ cancels; substitute; P and Q

(c) $\overrightarrow{PQ} = \begin{pmatrix}-2\\4\\3\end{pmatrix}$, $|\overrightarrow{PQ}| = \sqrt{(4+16+9)} = \sqrt{29}$ **M1 A1**

shortest distance: |PQ|