Natural Logarithms

Specification: AQA 7357: F3, F4, F7 · Edexcel 9MA0: 6.3, 6.4, 6.7

INDEPENDENT

You do

A · Exact solutions

1 Solve:
(a) $e^x = 12$
(b) $e^{2x} = 7$
(c) $e^{x-3} = 4$
(d) $4e^{3x} = 20$

(a) $x = \ln 12$ — take ln of both sides

(b) $2x = \ln 7 \Rightarrow x = \frac{1}{2}\ln 7$ — take ln; ÷ 2

(c) $x - 3 = \ln 4 \Rightarrow x = 3 + \ln 4$ — take ln; + 3

(d) $e^{3x} = 5 \Rightarrow 3x = \ln 5 \Rightarrow x = \frac{1}{3}\ln 5$ — ÷ 4 first; take ln

2 Solve:
(a) $\ln x = 5$
(b) $\ln(2x) = 3$
(c) $\ln(4x - 1) = 0$
(d) $3\ln x + 2 = 8$

(a) $x = e^5$ — e to the power of both sides

(b) $2x = e^3 \Rightarrow x = \frac{e^3}{2}$ — e to the power; ÷ 2

(c) $4x - 1 = e^0 = 1 \Rightarrow x = \frac{1}{2}$ — $e^0 = 1$

(d) $3\ln x = 6 \Rightarrow \ln x = 2 \Rightarrow x = e^2$ — make ln x the subject

3 Solve, to 3 s.f.:
(a) $e^{0.5x} = 30$
(b) $\ln(x + 4) = 1.5$

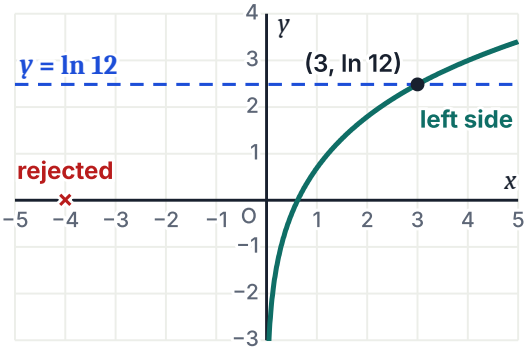
(a) $0.5x = \ln 30 \Rightarrow x = 2\ln 30 = 6.802\ldots$, so $x = 6.80$ (3 s.f.)
take ln; ÷ 0.5

(b) $x + 4 = e^{1.5} \Rightarrow x = e^{1.5} - 4 = 0.4816\ldots$, so $x = 0.482$ (3 s.f.)
e to the power; − 4

B · Laws and quadratics

4 Solve $\ln x + \ln(x + 1) = \ln 12$.

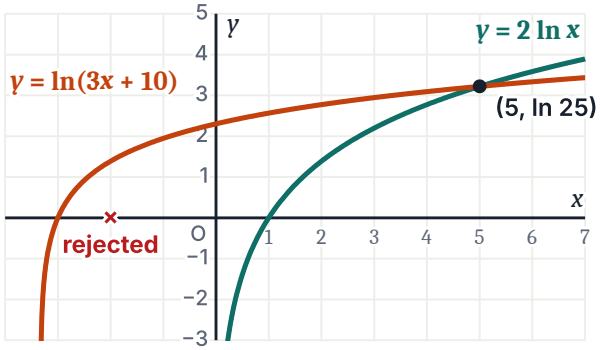
$\ln(x(x + 1)) = \ln 12 \Rightarrow x(x + 1) = 12$ — multiplication law
$x^2 + x - 12 = 0 \Rightarrow (x + 4)(x - 3) = 0$ — factorise
$x = 3$; reject $x = -4$ ($\ln(-4)$ is undefined)
check every log is defined



x = 3; x = −4 rejected: ln(−4) is undefined

5 Solve $2\ln x = \ln(3x + 10)$.

$\ln x^2 = \ln(3x + 10) \Rightarrow x^2 = 3x + 10$ — power law
$x^2 - 3x - 10 = 0 \Rightarrow (x - 5)(x + 2) = 0$ — factorise
$x = 5$; reject $x = -2$ ($\ln(-2)$ is undefined)
check every log is defined



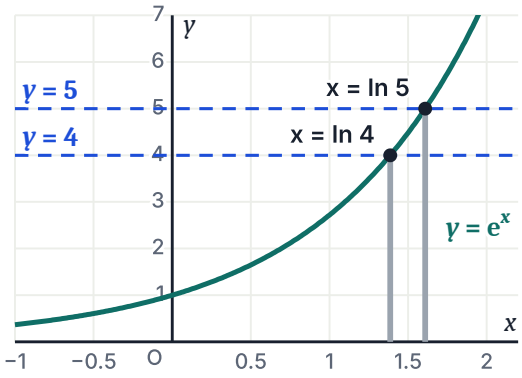
the curves meet once, at x = 5; x = −2 rejected (2 ln x needs x > 0)

6 Solve $e^{2x} - 9e^x + 20 = 0$.

Let $y = e^x$: $y^2 - 9y + 20 = (y-4)(y-5) = 0$ — let y = eˣ; factorise

$e^x = 4 \Rightarrow x = \ln 4$ — take ln of both sides

$e^x = 5 \Rightarrow x = \ln 5$ — take ln of both sides



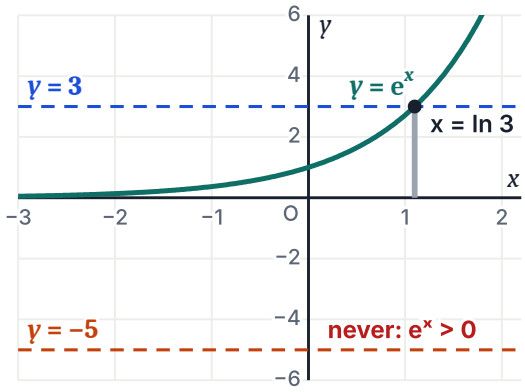
$e^x = 4$ at $x = \ln 4$; $e^x = 5$ at $x = \ln 5$

7 Solve $e^{2x} + 2e^x - 15 = 0$.

$(e^x + 5)(e^x - 3) = 0$ — let y = eˣ; factorise

$e^x = 3 \Rightarrow x = \ln 3$ — take ln of both sides

$e^x = -5$: no solution, because $e^x > 0$ — reject: eˣ > 0



$e^x = 3$ at $x = \ln 3$; e^x never reaches -5

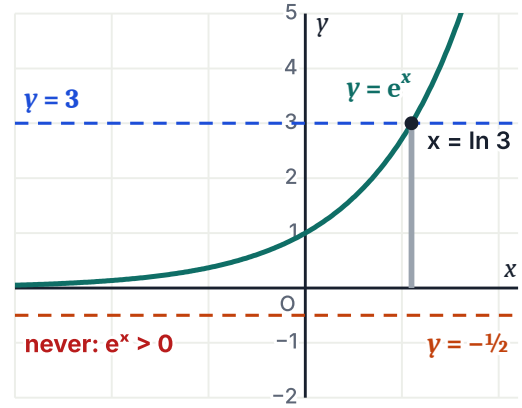
8 Solve $2e^x - 3e^{-x} = 5$.

$2e^{2x} - 3 = 5e^x \Rightarrow 2e^{2x} - 5e^x - 3 = 0$ — × eˣ; rearrange to = 0

$(2e^x + 1)(e^x - 3) = 0$ — factorise

$e^x = 3 \Rightarrow x = \ln 3$ — take ln of both sides

$e^x = -\frac{1}{2}$: no solution, because $e^x > 0$ — reject: eˣ > 0



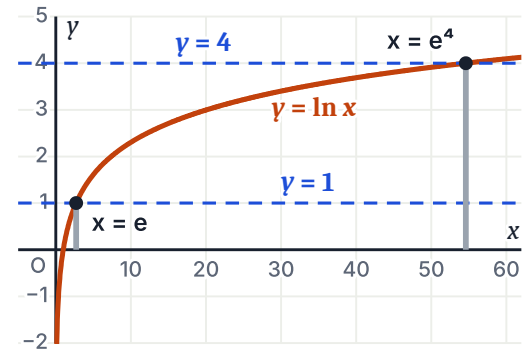
$e^x = 3$ at $x = \ln 3$; $e^x = -\frac{1}{2}$ is impossible: rejected

9 Solve $(\ln x)^2 - 5\ln x + 4 = 0$.

$(\ln x - 1)(\ln x - 4) = 0$ — let y = ln x; factorise

$\ln x = 1 \Rightarrow x = e$ — e to the power of both sides

$\ln x = 4 \Rightarrow x = e^4$ — e to the power of both sides



ln x = 1 at x = e; ln x = 4 at x = e⁴

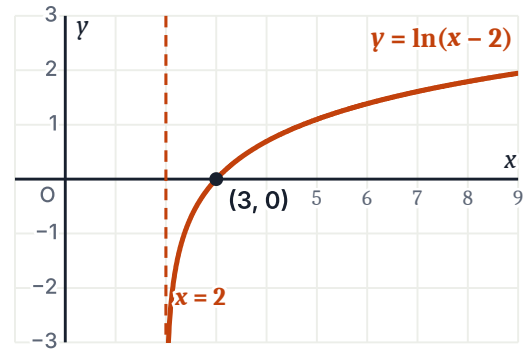
C · Graphs and reasoning

10 Sketch $y = \ln(x - 2)$. State the asymptote and where it crosses the x-axis.

Translation 2 right: asymptote $x = 2$ — f(x + a): opposite way

$y = 0$: $x - 2 = 1 \Rightarrow (3, 0)$ — x-intercept: ln 1 = 0

No y-intercept: $\ln(-2)$ does not exist. — x = 0 is left of x = 2



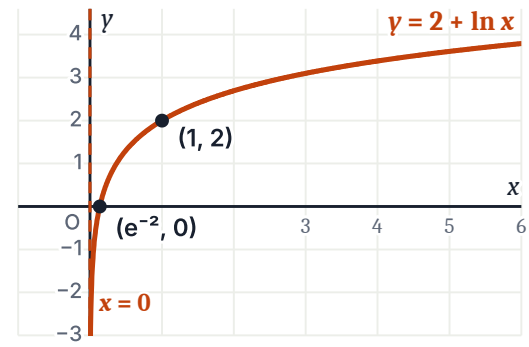
asymptote $x = 2$; crosses the x-axis at $(3, 0)$

11 Sketch $y = 2 + \ln x$. Find the exact x-intercept.

Translation 2 up: asymptote $x = 0$ — f(x) + a: up a

$y = 0$: $\ln x = -2$ — x-intercept: put y = 0

$x = e^{-2}$, so $(e^{-2}, 0)$ — e to the power of both sides



asymptote $x = 0$; x-intercept $(e^{-2}, 0)$

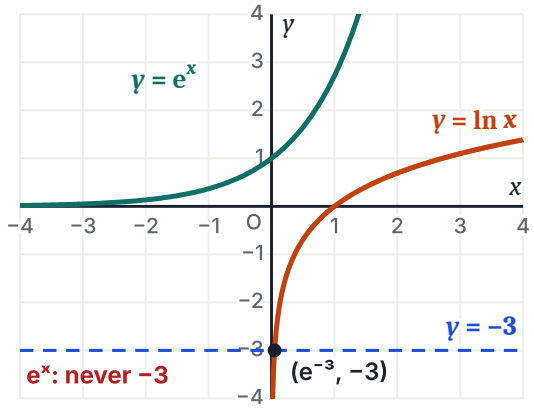
12 Explain why $\ln x = -3$ has a solution but $e^x = -3$ does not.

$\ln x$ takes every real value, so $x = e^{-3} > 0$ works.

ln: every real value

$e^x > 0$ for every x, so it can never be -3.

$e^x > 0$ always



ln x reaches −3 at x = e⁻³; eˣ > 0 never does

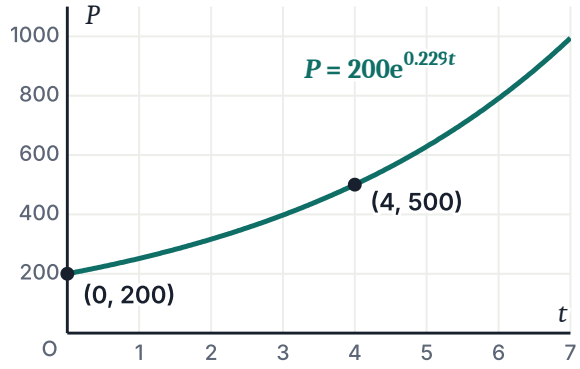
D · Modelling with ln

13 A population is modelled by $P = 200e^{kt}$. When $t = 4$, $P = 500$. Find k exactly and to 3 s.f.

$500 = 200e^{4k} \Rightarrow e^{4k} = 2.5$ — substitute; ÷ 200

$4k = \ln 2.5$ — take ln of both sides

$k = \dfrac{\ln 2.5}{4} = 0.2290\ldots$, so $k = 0.229$ (3 s.f.) — ÷ 4; round to 3 s.f.



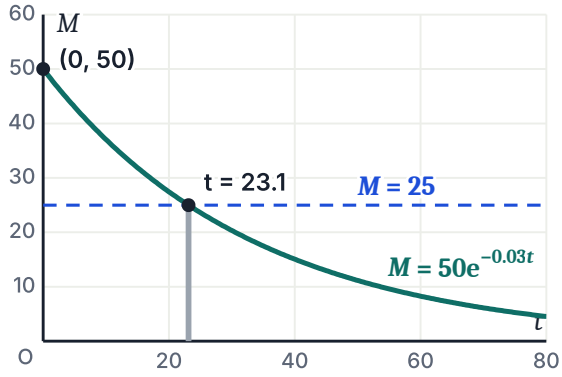
the model through (0, 200) and (4, 500): k = 0.229

14 A substance decays according to $M = 50e^{-0.03t}$, where t is in years. Find its half-life, to 3 s.f.

$25 = 50e^{-0.03t} \Rightarrow e^{-0.03t} = \dfrac{1}{2}$ — half of 50; ÷ 50

$-0.03t = \ln \dfrac{1}{2} = -\ln 2$ — take ln of both sides

$t = \dfrac{\ln 2}{0.03} = 23.10\ldots$, so 23.1 years (3 s.f.) — ÷ (−0.03); in context



half of 50 is 25: t = 23.1 years

E · Exam-style question AQA-style. Use the mark scheme to self-assess.

15 The concentration of a drug in the blood, C mg/l, t hours after a dose is modelled by $C = 8e^{-0.4t}$.

(a) State the initial concentration. [1 mark]

(b) Find the time when the concentration has fallen to 2 mg/l. Give an exact answer and an answer to 3 s.f. [3 marks]

(c) Show that $\frac{dC}{dt} = -0.4C$ and interpret the sign. [2 marks]

(a) $t = 0$: $C = 8e^0 = 8$, so 8 mg/l **B1** — initial value: t = 0

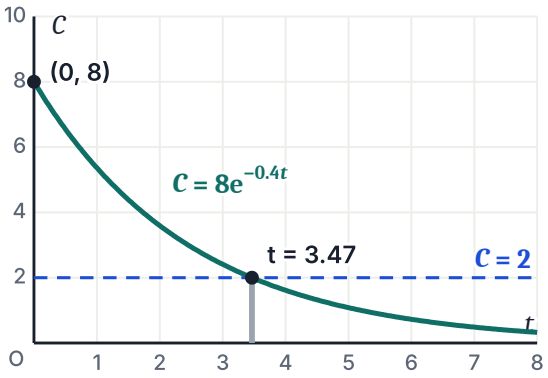
(b) $2 = 8e^{-0.4t} \Rightarrow e^{-0.4t} = \frac{1}{4}$ **M1** — substitute; ÷ 8

$-0.4t = \ln\frac{1}{4} = -\ln 4 \Rightarrow t = \frac{\ln 4}{0.4} = 2.5 \ln 4$ **M1** — take ln; ÷ (−0.4)

$t = 3.465...$, so $t = 3.47$ hours (3 s.f.) **A1** — calculator, then round

(c) $\frac{dC}{dt} = 8 \times (-0.4)e^{-0.4t} = -0.4 \times 8e^{-0.4t} = -0.4C$ **B1** — differentiate: × k

Negative: the concentration is decreasing. **E1** — in context



(a) C = 8 at t = 0 (b) C = 2 at t = 3.47