Locating Roots

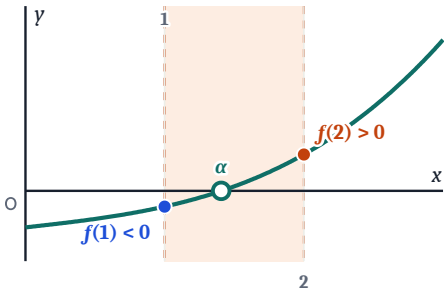
Specification: AQA 7357: I1 · Edexcel 9MA0: 9.1

INDEPENDENT **You do**

A · Showing a root Show that $f(x) = 0$ has a root in the given interval.

1 $f(x) = x^3 + 3x - 7,\ 1 < x < 2$

$f(1) = -3 < 0,\ f(2) = 7 > 0$ — substitute both ends



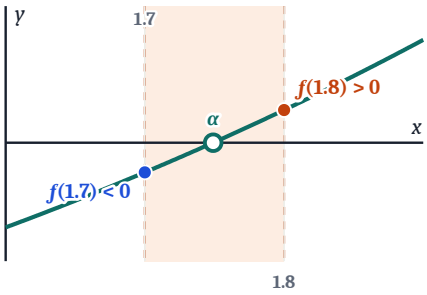
change of sign: a root

Change of sign, and f is continuous on $[1, 2]$, so there is a root $1 < x < 2$.

continuous ⇒ root

2 $f(x) = e^x - x - 4,\ 1.7 < x < 1.8$

$f(1.7) = -0.226 < 0,\ f(1.8) = 0.250 > 0$ — substitute both ends



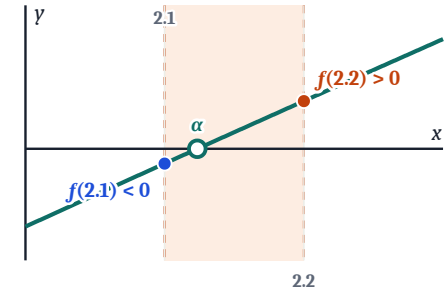
change of sign: a root

Change of sign, and f is continuous on $[1.7, 1.8]$, so there is a root $1.7 < x < 1.8$.

continuous ⇒ root

3 $f(x) = \ln x + 2x - 5,\ 2.1 < x < 2.2$

$f(2.1) = -0.0581 < 0,\ f(2.2) = 0.188 > 0$ — substitute both ends



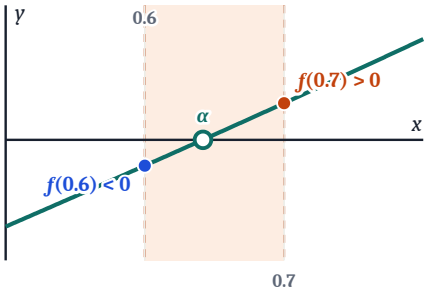
change of sign: a root

Change of sign, and f is continuous on $[2.1, 2.2]$, so there is a root $2.1 < x < 2.2$.

continuous ⇒ root

4 $f(x) = \sqrt{x} - \cos x$ (x in radians), $0.6 < x < 0.7$

$f(0.6) = -0.0507 < 0,\ f(0.7) = 0.0718 > 0$ — substitute both ends



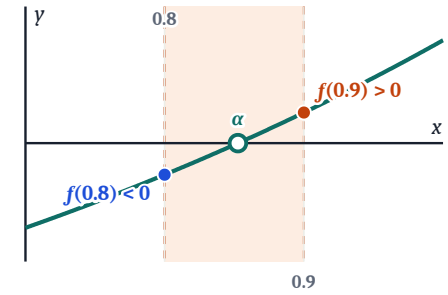
change of sign: a root

Change of sign, and f is continuous on $[0.6, 0.7]$, so there is a root $0.6 < x < 0.7$.

continuous ⇒ root

5 $f(x) = xe^x - 2,\ 0.8 < x < 0.9$

$f(0.8) = -0.220 < 0,\ f(0.9) = 0.214 > 0$ — substitute both ends



change of sign: a root

Change of sign, and f is continuous on $[0.8, 0.9]$, so there is a root $0.8 < x < 0.9$.

continuous ⇒ root

6 $f(x) = x^3 + x - 5$.

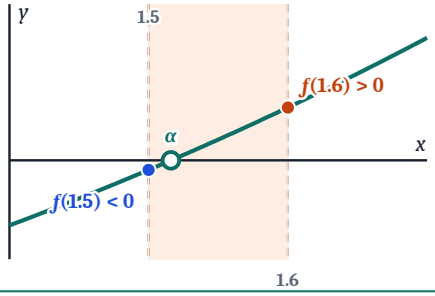
(a) Complete the table.

(b) Between which two consecutive values of x is the root of $f(x) = 0$?

x	1.3	1.4	1.5	1.6	1.7
$f(x)$	−1.503	−0.856	−0.125	0.696	1.613

(a) Substitute each x (click the cells to check). — substitute each x

(b) The sign changes from $f(1.5) = -0.125 < 0$ to $f(1.6) = 0.696 > 0$. — look for the sign change



sign changes between 1.5 and 1.6

f is continuous, so the root is between $x = 1.5$ and $x = 1.6$. — continuous ⇒ root

B · To a given accuracy Write down the bounds, substitute them, conclude.

7 Write down the two values of x you would substitute to show that:

(a) $\alpha = 2.47$ to 2 d.p.

(b) $\alpha = 0.836$ to 3 d.p.

(c) $\alpha = -1.3$ to 1 d.p.

(d) $\alpha = 1.070$ to 3 d.p.

(a) 2.465 and 2.475 — ± 0.005

(b) 0.8355 and 0.8365 — ± 0.0005

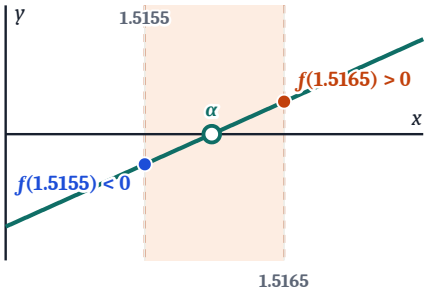
(c) −1.35 and −1.25 — ± 0.05

(d) 1.0695 and 1.0705 — ± 0.0005: keep the 0

8 The equation $x^3 + x - 5 = 0$ has a root α (question 6). Show that $\alpha = 1.516$ correct to 3 d.p.

Every number from 1.5155 up to (not including) 1.5165 rounds to 1.516. Test the bounds 1.5155 and 1.5165. — bounds: ±0.0005

$f(1.5155) = -0.00379 < 0$, $f(1.5165) = 0.00410 > 0$ — substitute both bounds



α between the bounds

Change of sign and f is continuous, so $1.5155 < \alpha < 1.5165$. So $\alpha = 1.516$ to 3 d.p. — continuous ⇒ root

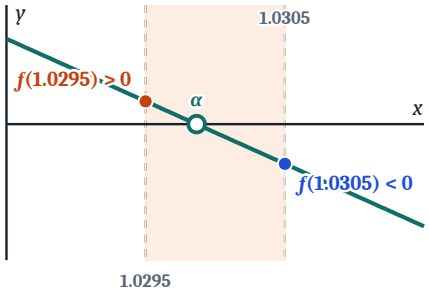
9 The equation $2\cos x - x = 0$ (x in radians) has a root α. Show that $\alpha = 1.030$ correct to 3 d.p.

Every number from 1.0295 up to (not including) 1.0305 rounds to 1.030. Test the bounds 1.0295 and 1.0305.

bounds: ±0.0005

$f(1.0295) = 0.000995 > 0, f(1.0305) = -0.00172 < 0$

substitute both bounds



α between the bounds

Change of sign and f is continuous, so $1.0295 < \alpha < 1.0305$. So $\alpha = 1.030$ to 3 d.p.

continuous ⇒ root

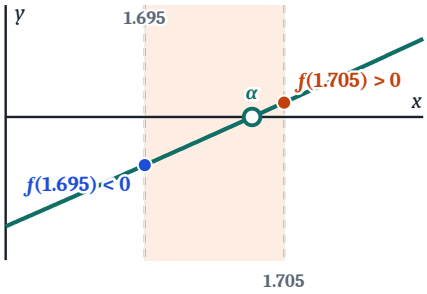
10 The equation $x^4 - 2x - 5 = 0$ has a root α between 1.7 and 1.8. Show that $\alpha = 1.70$ correct to 2 d.p.

Every number from 1.695 up to (not including) 1.705 rounds to 1.70. Test the bounds 1.695 and 1.705.

bounds: ±0.005

$f(1.695) = -0.136 < 0, f(1.705) = 0.0408 > 0$

substitute both bounds



α between the bounds

Change of sign and f is continuous, so $1.695 < \alpha < 1.705$. So $\alpha = 1.70$ to 2 d.p.

continuous ⇒ root

C · Reasoning Explain fully. Q11–Q13: no calculator.

11 $f(x) = \dfrac{3}{x-1} + 2,\ x \neq 1.$

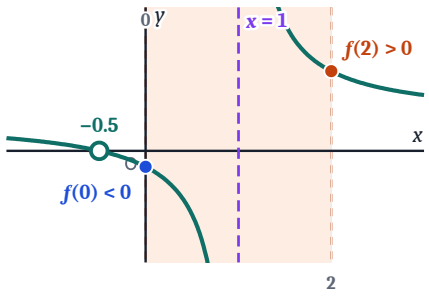
(a) Show that $f(0)$ and $f(2)$ have opposite signs.

(b) Solve $f(x) = 0$.

(c) Explain why $f(x) = 0$ has no root in $0 < x < 2$.

(a) $f(0) = -1 < 0, f(2) = 5 > 0$ substitute both ends

(b) $\dfrac{3}{x-1} = -2 \Rightarrow x - 1 = -1.5 \Rightarrow x = -0.5$ solve



asymptote x = 1; root x = −0.5

(c) The only root is $x = -0.5$. The sign changes at the asymptote $x = 1$: f is not continuous on $[0, 2]$.

asymptote: not continuous

12 $f(x) = 4x^2 - 12x + 9.$

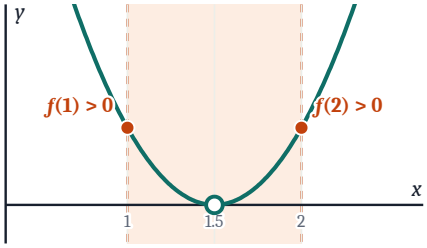
(a) Find $f(1)$ and $f(2)$.

(b) Factorise $f(x)$.

(c) Explain why a change of sign can never locate the root of $f(x) = 0$.

(a) $f(1) = 1 > 0, f(2) = 1 > 0$ substitute both ends

(b) $f(x) = (2x-3)^2$ perfect square



touches at x = 1.5

(c) Repeated root $x = 1.5$. $(2x-3)^2 \geq 0$: the curve touches the axis and never goes below it, so f never changes sign.

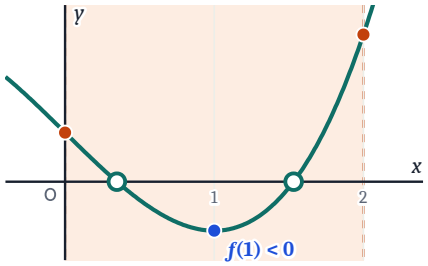
repeated root: touches

13 $f(x) = x^3 - 3x + 1$.
(a) Show that $f(0)$ and $f(2)$ have the same sign.
(b) By finding $f(1)$, show that $f(x) = 0$ has at least two roots in $0 < x < 2$.

(a) $f(0) = 1 > 0$, $f(2) = 3 > 0$: no change of sign.

substitute both ends

(b) $f(1) = -1 < 0$ substitute x = 1



two roots in (0, 2)

Change of sign on $[0, 1]$ and on $[1, 2]$, f continuous: at least two roots in $0 < x < 2$.

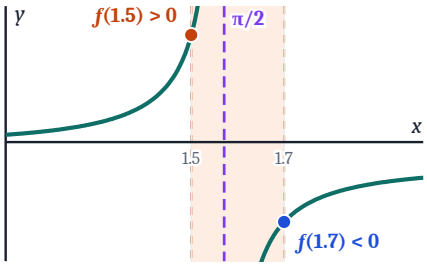
even number of roots

14 $f(x) = \tan x - x$ (x in radians).
(a) Show that $f(1.5)$ and $f(1.7)$ have opposite signs.
(b) Explain why this does not show that $f(x) = 0$ has a root in $1.5 < x < 1.7$.

(a) $f(1.5) = 12.6 > 0$, $f(1.7) = -9.40 < 0$ calculator: radians

(b) $\tan x$ has an asymptote at $x = \frac{\pi}{2} \approx 1.571$, inside $[1.5, 1.7]$.

asymptote at π/2



asymptote π/2: no root

f is not continuous on $[1.5, 1.7]$, so the change of sign proves nothing (here there is no root).

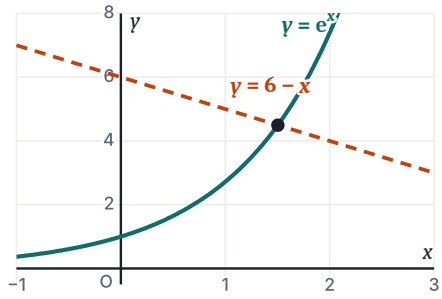
not continuous

D · Show that

15 (a) On the same axes, sketch $y = e^x$ and $y = 6 - x$.
(b) Explain why $e^x + x - 6 = 0$ has exactly one root, α.
(c) Show that $1.5 < \alpha < 1.6$.
(d) Show that $\alpha = 1.503$ correct to 3 d.p.

(a) $y = e^x$: through $(0, 1)$, increasing. $y = 6 - x$: line through $(0, 6)$ and $(6, 0)$.

sketch



one crossing: one root

(b) $e^x = 6 - x$: the graphs cross once, so exactly one root.

count intersections

(c) $f(1.5) = -0.0183 < 0$, $f(1.6) = 0.553 > 0$: change of sign, f continuous, so $1.5 < \alpha < 1.6$.

continuous ⇒ root

(d) Every number from 1.5025 up to (not including) 1.5035 rounds to 1.503. Test the bounds 1.5025 and 1.5035.

bounds: ±0.0005

$f(1.5025) = -0.00459 < 0$, $f(1.5035) = 0.000902 > 0$

substitute both bounds

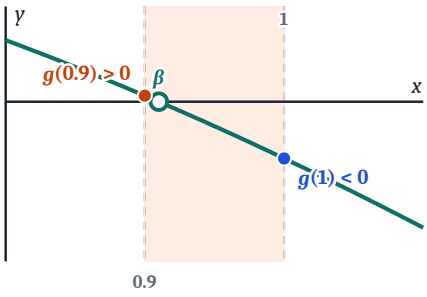
Change of sign and f is continuous, so $1.5025 < \alpha < 1.5035$. So $\alpha = 1.503$ to 3 d.p.

continuous ⇒ root

16 (a) Show that the curve $y = e^x - x^3$ has a stationary point with x-coordinate β between 0.9 and 1.
(b) Show that $\beta = 0.91$ correct to 2 d.p.

(a) $\frac{dy}{dx} = e^x - 3x^2$. Let $g(x) = e^x - 3x^2$. differentiate

$g(0.9) = 0.0296 > 0$, $g(1) = -0.282 < 0$ substitute both ends



dy/dx changes sign

Change of sign, g continuous: $\frac{dy}{dx} = 0$ for some $0.9 < \beta < 1$.

f' changes sign: stationary point

(b) Every number from 0.905 up to (not including) 0.915 rounds to 0.91. Test the bounds 0.905 and 0.915.

bounds: ±0.005

$g(0.905) = 0.0149 > 0$, $g(0.915) = -0.0149 < 0$

substitute both bounds

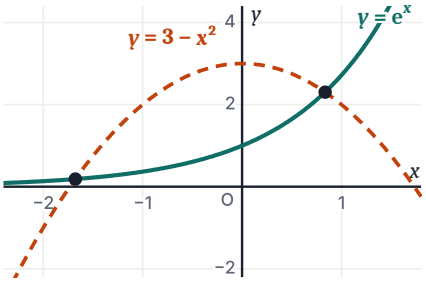
Change of sign and g is continuous, so $0.905 < \beta < 0.915$. So $\beta = 0.91$ to 2 d.p.

continuous ⇒ root

17 (a) On the same axes, sketch $y = e^x$ and $y = 3 - x^2$.
(b) State the number of roots of $e^x + x^2 - 3 = 0$.
(c) Show that the negative root lies between -1.7 and -1.6.

(a) $y = e^x$: through $(0, 1)$. $y = 3 - x^2$: ∩-shaped, through $(0, 3)$ and $(\pm\sqrt{3}, 0)$.

sketch



two crossings: two roots

(b) The graphs cross twice: **two** roots. count intersections

(c) $f(-1.7) = 0.0727 > 0$, $f(-1.6) = -0.238 < 0$

substitute both ends

Change of sign, and f is continuous on $[-1.7, -1.6]$, so there is a root $-1.7 < x < -1.6$.

continuous ⇒ root

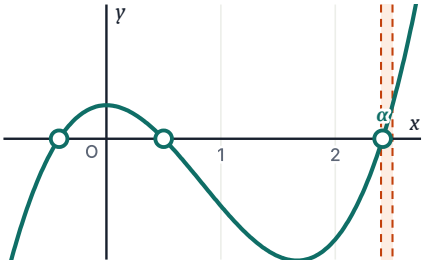
E · Exam-style question AQA-style. Use the mark scheme to self-assess.

18 $f(x) = 2x^3 - 5x^2 + 1$.
(a) Show that $f(x) = 0$ has a root α in the interval $[2.4, 2.5]$. [2 marks]
(b) Show that $\alpha = 2.414$ correct to 3 decimal places. [3 marks]
(c) Show that $(2x - 1)$ is a factor of $f(x)$. Hence find the exact value of α. [3 marks]

(a) $f(2.4) = -0.152 < 0$, $f(2.5) = 1 > 0$ **M1** substitute both ends
Change of sign and f is continuous, so $2.4 < \alpha < 2.5$. **A1** continuous ⇒ root

(b) Bounds 2.4135 and 2.4145: $f(2.4135) = -0.00772 < 0$ **M1** bounds: ±0.0005
$f(2.4145) = 0.00310 > 0$ **A1** substitute both bounds
Change of sign, continuous, so $2.4135 < \alpha < 2.4145$: $\alpha = 2.414$ to 3 d.p. **A1** continuous ⇒ root

(c) $f(\frac{1}{2}) = \frac{2}{8} - \frac{5}{4} + 1 = 0$, so $(2x - 1)$ is a factor. **B1** factor theorem: f(½) = 0
$f(x) = (2x - 1)(x^2 - 2x - 1)$ **M1** divide by (2x − 1)
$x^2 - 2x - 1 = 0 \Rightarrow x = 1 \pm \sqrt{2}$. α is in $[2.4, 2.5]$, so $\alpha = 1 + \sqrt{2}$. **A1** solve the quadratic; choose α



three roots: α = 1 + √2