Iteration

Specification: AQA 7357: I2 · Edexcel 9MA0: 9.2

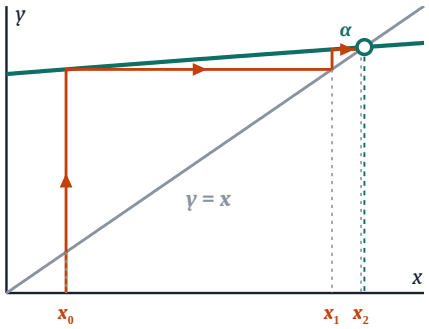
INDEPENDENT **You do**

A · Iterating Find x_1, x_2 and x_3 to 4 decimal places.

1 $x_{n+1} = \sqrt[3]{x_n + 4}$, $x_0 = 1$

$x_1 = \sqrt[3]{1 + 4} = 1.7100$ — substitute x_0

$x_2 = 1.7874$, $x_3 = 1.7954$ — ANS key: full value

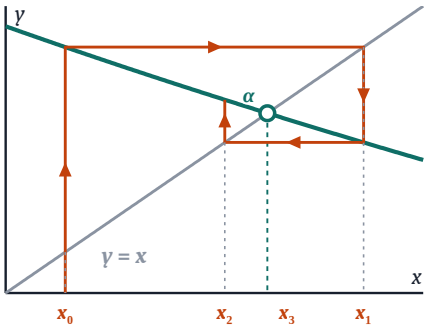


converges

2 $x_{n+1} = 3 - \ln x_n$, $x_0 = 2$

$x_1 = 3 - \ln 2 = 2.3069$ — substitute x_0

$x_2 = 2.1641$, $x_3 = 2.2280$ — ANS key: full value

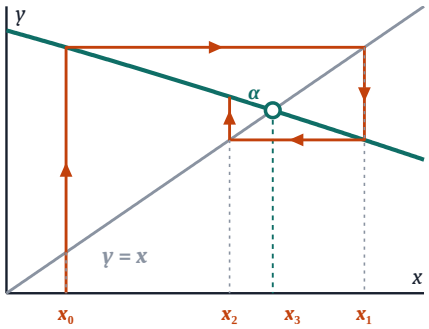


converges: up and down

3 $x_{n+1} = 1 + \frac{1}{2}\cos x_n$ (x_n in radians), $x_0 = 1$

$x_1 = 1 + \frac{1}{2}\cos 1 = 1.2702$ — substitute x_0 (radians)

$x_2 = 1.1481$, $x_3 = 1.2051$ — ANS key (radians)

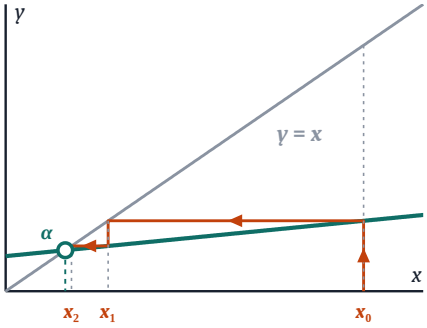


converges: up and down

4 $x_{n+1} = \ln(x_n + 5)$, $x_0 = 2$

$x_1 = \ln(2 + 5) = 1.9459$ — substitute x_0

$x_2 = 1.9382$, $x_3 = 1.9370$ — ANS key: full value

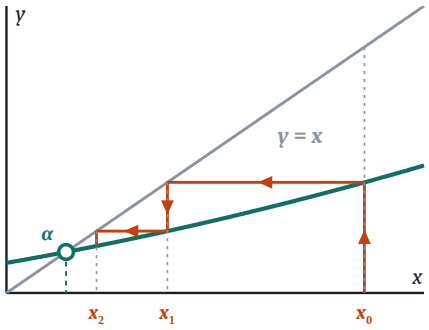


converges

5 $x_{n+1} = \dfrac{x_n^{\,2} + 3}{5}$, $x_0 = 1$

$x_1 = \dfrac{1^2 + 3}{5} = 0.8000$ substitute x_0

$x_2 = 0.7280$, $x_3 = 0.7060$ ANS key: full value



converges

6 (a) Use $x_{n+1} = \sqrt[3]{10 - 2x_n}$ with $x_0 = 2$ to complete the table (4 d.p.).
(b) Describe how the values approach the root.

n	0	1	2	3	4
x_n	2	1.8171	1.8533	1.8463	1.8476

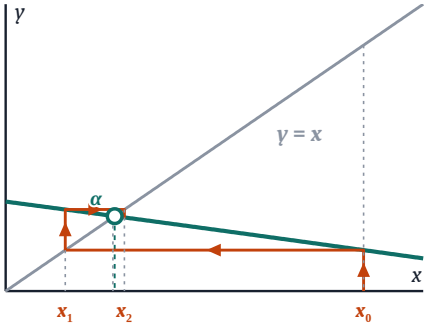
(a) $x_1 = \sqrt[3]{10 - 2 \times 2} = 1.8171$ substitute x_0

Carry on with the ANS key (click the cells to check x_2, x_3, x_4).

ANS key: full value

(b) The values go below, above, below, ... and get closer each time: they converge to 1.8474.

converges: up and down



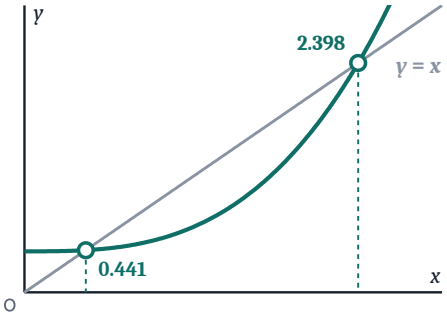
converges: up and down

B · Rearranging Show that each equation can be written in the form given.

7 $x^3 - 7x + 3 = 0$ as $x = \dfrac{x^3 + 3}{7}$

$7x = x^3 + 3$ 7x to one side

$x = \dfrac{x^3 + 3}{7}$ ÷ 7



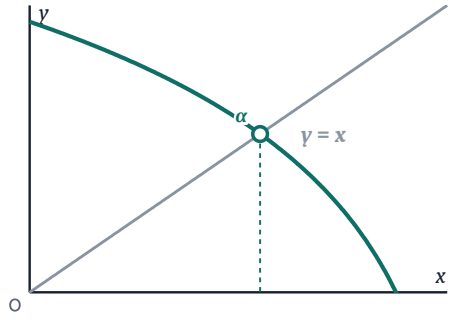
y = x meets y = g(x) at the roots

8 $e^x + 3x - 6 = 0$ as $x = \ln(6 - 3x)$, $x < 2$

$e^x = 6 - 3x$ 3x and 6 to the right

$x = \ln(6 - 3x)$ (needs $6 - 3x > 0$, so $x < 2$)

take ln of both sides



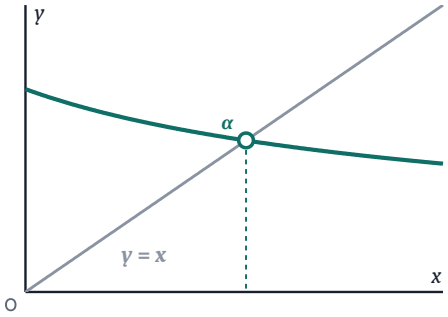
y = x meets y = g(x) at the root

9 $x^3 + 2x^2 - 9 = 0$ as $x = \dfrac{3}{\sqrt{x+2}}$, $x > 0$

$x^2(x+2) = 9$ — factorise x^2

$x^2 = \dfrac{9}{x+2}$ — ÷ (x + 2)

$x = \dfrac{3}{\sqrt{x+2}}$ — square root: x > 0



y = x meets y = g(x) at the root

10 $x^3 - 2x^2 - 4 = 0$ as:

(a) $x = 2 + \dfrac{4}{x^2}$

(b) $x = \dfrac{2}{\sqrt{x-2}}$, $x > 2$

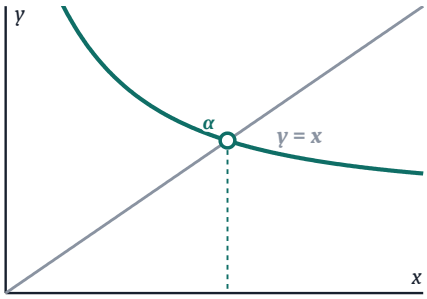
(a) $x^3 = 2x^2 + 4$ — $2x^2$ and 4 to the right

$x = 2 + \dfrac{4}{x^2}$ $(x \neq 0)$ — ÷ x^2

(b) $x^2(x-2) = 4$ — factorise x^2

$x^2 = \dfrac{4}{x-2}$ $(x \neq 2)$ — ÷ (x − 2)

$x = \dfrac{2}{\sqrt{x-2}}$ — square root: x > 2



(a): y = x meets y = 2 + 4/x²

C · Reasoning Explain fully. Q11 and Q12: no calculator.

11 $x_{n+1} = \dfrac{1}{2}\left(x_n + \dfrac{5}{x_n}\right)$, $x_0 = 2$.

(a) Find x_1 as a fraction.

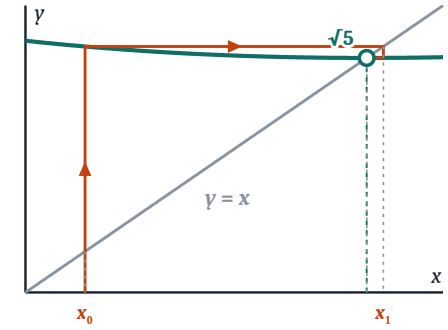
(b) The iterates converge to α. Show that $\alpha^2 = 5$ and write down the exact value of α.

(a) $x_1 = \dfrac{1}{2}\left(2 + \dfrac{5}{2}\right) = \dfrac{1}{2} \times \dfrac{9}{2} = \dfrac{9}{4}$ — substitute x_0

(b) $\alpha = \dfrac{1}{2}\left(\alpha + \dfrac{5}{\alpha}\right)$ — limit: α = g(α)

$2\alpha = \alpha + \dfrac{5}{\alpha} \Rightarrow \alpha = \dfrac{5}{\alpha} \Rightarrow \alpha^2 = 5$ — rearrange

$\alpha = \sqrt{5}$ (every iterate is positive) — positive root



converges to √5

12 $x_{n+1} = 4 - \dfrac{3}{x_n}$, $x_0 = 2$.

(a) Find x_1 and x_2.

(b) The iterates converge to α. Show that $\alpha^2 - 4\alpha + 3 = 0$.

(c) Solve this equation. Which solution is α?

(a) $x_1 = 4 - \dfrac{3}{2} = 2.5$ — substitute x_0

$x_2 = 4 - \dfrac{3}{2.5} = 4 - 1.2 = 2.8$ — substitute x_1

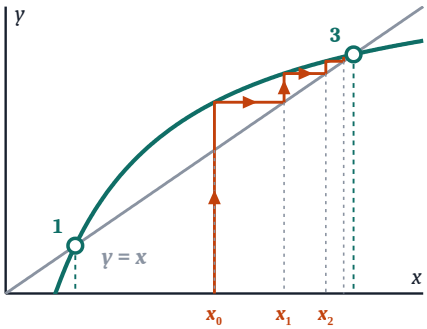
(b) $\alpha = 4 - \dfrac{3}{\alpha} \Rightarrow \alpha^2 = 4\alpha - 3 \Rightarrow \alpha^2 - 4\alpha + 3 = 0$

limit: α = g(α)

(c) $(\alpha - 1)(\alpha - 3) = 0 \Rightarrow \alpha = 1$ or $\alpha = 3$ — factorise

The values 2, 2.5, 2.8, … are increasing past 2, towards 3: $\alpha = 3$.

which root?



converges to 3, not 1

13 $x_{n+1} = \sqrt{7 - 4x_n}$.

(a) Explain why $x_0 = 2$ cannot be used.

(b) With $x_0 = 1$, find x_1, x_2, x_3.

(c) What happens when you try to find x_4?

(a) $x_1 = \sqrt{7 - 4 \times 2} = \sqrt{-1}$: not a real number.

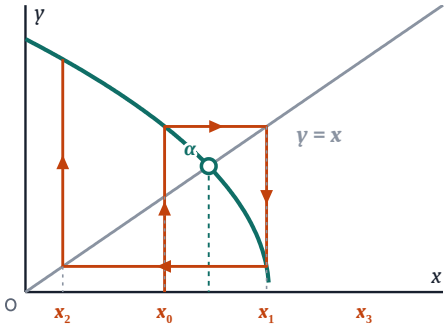
√ of a negative

(b) $x_1 = \sqrt{7 - 4 \times 1} = 1.7321$ substitute x_0

$x_2 = 0.2679$, $x_3 = 2.4348$ ANS key: full value

(c) $x_4 = \sqrt{7 - 4 \times 2.4348\ldots} = \sqrt{-2.7392\ldots}$: not real. The values jumped further and further from the root, and the iteration **fails**.

fails: √ of a negative



jumps away, then fails

14 $x^3 + x - 4 = 0$ has one root α, $1.3 < \alpha < 1.4$.

(a) Show that it can be written as $x = \dfrac{4}{x^2 + 1}$.

(b) Use $x_{n+1} = \dfrac{4}{x_n^{\,2} + 1}$ with $x_0 = 1.4$ to find x_1 to x_4.

(c) Does the iteration converge to α? Explain.

(a) $x^3 + x = 4$ 4 to the right

$x(x^2 + 1) = 4$ factorise x

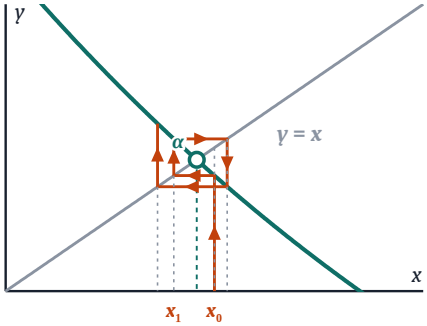
$x = \dfrac{4}{x^2 + 1}$ ÷ (x² + 1)

(b) $x_1 = \dfrac{4}{(1.4)^2 + 1} = 1.3514$ substitute x_0

$x_2 = 1.4154$, $x_3 = 1.3319$, $x_4 = 1.4420$ ANS key: full value

(c) No. The values jump either side of $\alpha = 1.3788$ and move further away each time: it diverges.

diverges: up and down



jumps away: diverges

D · Show that

15 $f(x) = x^3 - 3x^2 - 5$ has one root, α.

(a) Show that $f(x) = 0$ can be written as $x = 3 + \dfrac{5}{x^2}$.

(b) Use $x_{n+1} = 3 + \dfrac{5}{x_n^{\,2}}$ with $x_0 = 3.4$ to find x_1, x_2, x_3.

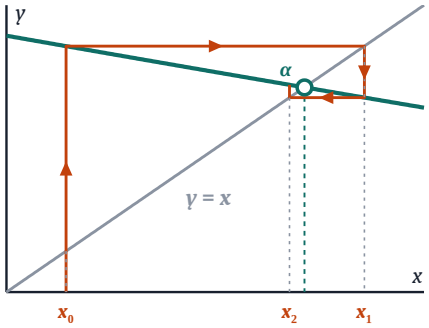
(c) Show that $\alpha = 3.426$ correct to 3 d.p.

(a) $x^2(x - 3) = 5 \Rightarrow x - 3 = \dfrac{5}{x^2} \Rightarrow x = 3 + \dfrac{5}{x^2}$

factorise x^2, $\div x^2$

(b) $x_1 = 3 + \dfrac{5}{(3.4)^2} = 3.4325$ substitute x_0

$x_2 = 3.4244$, $x_3 = 3.4264$ ANS key: full value



x_1, x_2, x_3 close in on α

(c) Bounds 3.4255 and 3.4265: $f(3.4255) = -0.00716 < 0$, $f(3.4265) = 0.00749 > 0$

bounds: ±0.0005

Change of sign and f is continuous, so $3.4255 < \alpha < 3.4265$. So $\alpha = 3.426$ to 3 d.p.

continuous ⇒ root

16 $f(x) = \ln x + 2x - 5$ has a root α between 2.1 and 2.2.

(a) Show that $f(x) = 0$ can be written as $x = \dfrac{5 - \ln x}{2}$.

(b) Use $x_{n+1} = \dfrac{5 - \ln x_n}{2}$ with $x_0 = 2$ to find x_1, x_2, x_3.

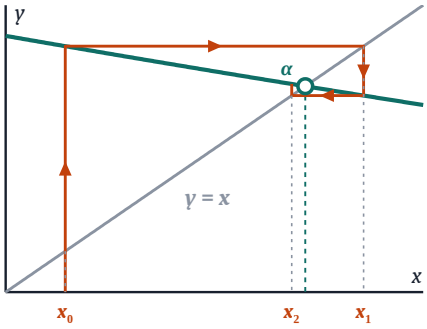
(c) Show that $\alpha = 2.12$ correct to 2 d.p.

(a) $2x = 5 - \ln x$ ln x and 5 to the right

$x = \dfrac{5 - \ln x}{2}$ $\div 2$

(b) $x_1 = \dfrac{5 - \ln 2}{2} = 2.1534$ substitute x_0

$x_2 = 2.1165$, $x_3 = 2.1251$ ANS key: full value



x_1, x_2, x_3 close in on α

(c) Bounds 2.115 and 2.125: $f(2.115) = -0.0209 < 0$, $f(2.125) = 0.00377 > 0$

bounds: ±0.005

Change of sign and f is continuous, so $2.115 < \alpha < 2.125$. So $\alpha = 2.12$ to 2 d.p.

continuous ⇒ root

17 The curve $y = x^2 + 4\cos x$ (x in radians) has a stationary point at $x = \beta$, $1.8 < \beta < 1.9$.

(a) Show that β satisfies $x = 2\sin x$.

(b) Use $x_{n+1} = 2\sin x_n$ with $x_0 = 2$ to find x_1, x_2, x_3.

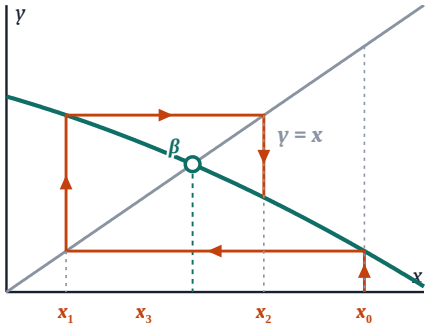
(c) Show that $\beta = 1.90$ correct to 2 d.p.

(a) $\dfrac{dy}{dx} = 2x - 4\sin x$. Let $h(x) = 2x - 4\sin x$. — differentiate

$2x - 4\sin x = 0 \Rightarrow x = 2\sin x$ — stationary: dy/dx = 0

(b) $x_1 = 2\sin 2 = 1.8186$ — substitute x_0 (radians)

$x_2 = 1.9389$, $x_3 = 1.8660$ — ANS key (radians)



converges: up and down

(c) Bounds 1.895 and 1.905: $h(1.895) = -0.00162 < 0$, $h(1.905) = 0.0313 > 0$ — bounds: ±0.005

Change of sign and h is continuous, so $1.895 < \beta < 1.905$. So $\beta = 1.90$ to 2 d.p. — continuous ⇒ root

E · Exam-style question AQA-style (AQA starts from x_1). Use the mark scheme to self-assess.

18 $f(x) = x^2 - 3 - \ln(x+1)$, $x > -1$.

(a) Show that $f(x) = 0$ has a root α between 2 and 2.1. [2 marks]

(b) Show that $f(x) = 0$ can be rearranged into the form $x = \sqrt{3 + \ln(x+1)}$. [2 marks]

(c) Use $x_{n+1} = \sqrt{3 + \ln(x_n + 1)}$ with $x_1 = 1.5$ to find x_2 and x_3, giving your answers to three decimal places. [3 marks]

(d) Explain why $x_1 = -1$ cannot be used. [1 mark]

(a) $f(2) = -0.0986 < 0$, $f(2.1) = 0.279 > 0$ **M1** — substitute both ends

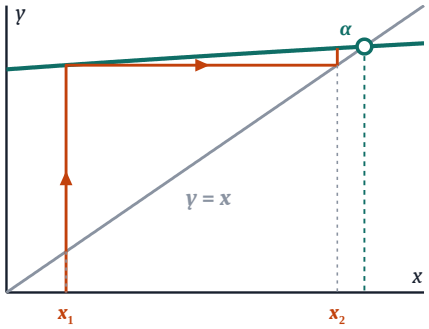
Change of sign and f is continuous for $x > -1$, so $2 < \alpha < 2.1$. **A1** — continuous ⇒ root

(b) $x^2 = 3 + \ln(x+1)$ **M1** — x^2 to one side

$x = \sqrt{3 + \ln(x+1)}$ (positive root: $\alpha > 0$) **A1** — square root: x > 0

(c) $x_2 = \sqrt{3 + \ln(1.5+1)} = 1.979$ **M1 A1** — substitute x_1

$x_3 = \sqrt{3 + \ln(1.979\ldots + 1)} = 2.023$ **A1** — ANS key: full value



x_1, x_2, x_3 close in on α

(d) $\ln(-1+1) = \ln 0$ is not defined, so x_2 cannot be found. **B1** — ln 0: not defined