Staircase and Cobweb Diagrams

Specification: AQA 7357: I2 · Edexcel 9MA0: 9.2

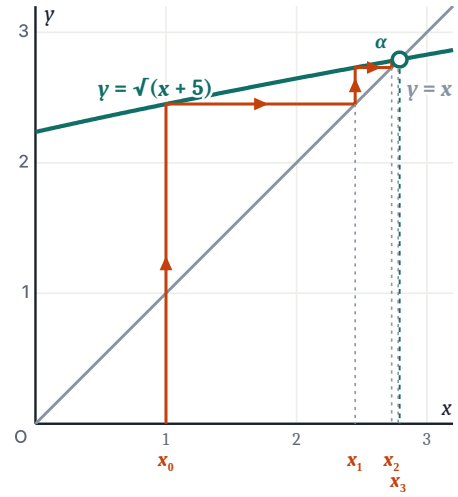
INDEPENDENT **You do**

A · Staircase or cobweb? Find x_1, x_2 and x_3 to 4 decimal places. Would the diagram be a staircase or a cobweb? Does it converge?

1 $x_{n+1} = \sqrt{x_n + 5}$, $x_0 = 1$

$x_1 = \sqrt{1+5} = 2.4495$ substitute x_0

$x_2 = 2.7294$, $x_3 = 2.7802$ ANS key: full value



staircase

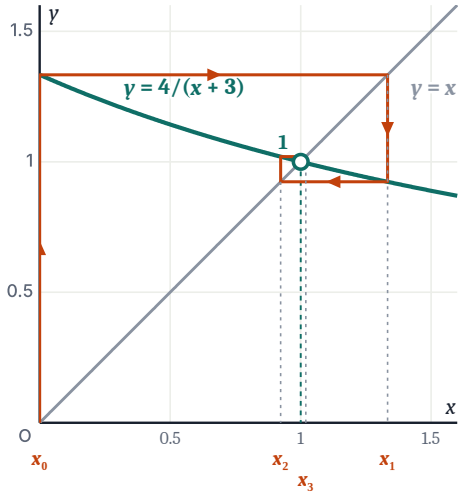
Increasing, smaller steps: a **staircase** that converges (to 2.7913).

one way: staircase

2 $x_{n+1} = \dfrac{4}{x_n + 3}$, $x_0 = 0$

$x_1 = \dfrac{4}{0+3} = 1.3333$ substitute x_0

$x_2 = 0.9231$, $x_3 = 1.0196$ ANS key: full value



cobweb

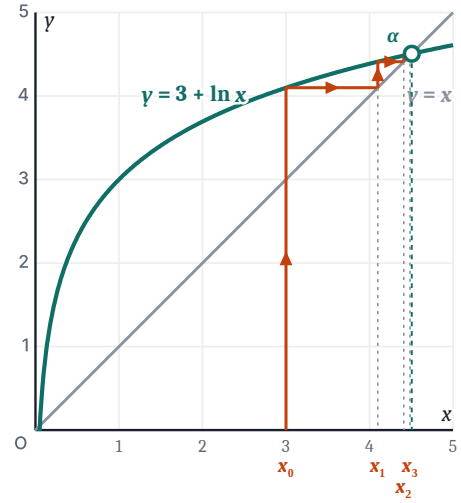
Above, below, above 1, closer each time: a **cobweb** that converges (to 1).

either side: cobweb

3 $x_{n+1} = 3 + \ln x_n$, $x_0 = 3$

$x_1 = 3 + \ln 3 = 4.0986$ substitute x_0

$x_2 = 4.4106$, $x_3 = 4.4840$ ANS key: full value



staircase

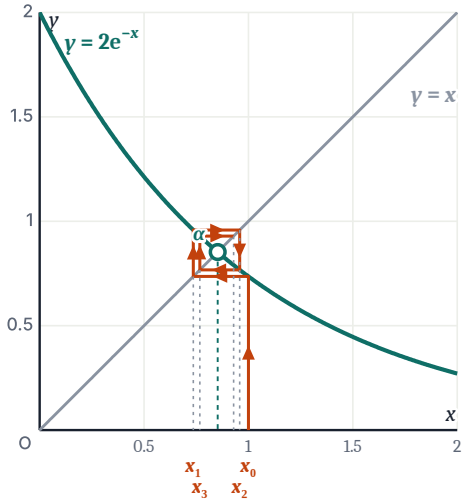
Increasing, smaller steps: a **staircase** that converges (to 4.5052).

one way: staircase

4 $x_{n+1} = 2e^{-x_n}$, $x_0 = 1$

$x_1 = 2e^{-1} = 0.7358$ substitute x_0

$x_2 = 0.9583$, $x_3 = 0.7671$ ANS key: full value



cobweb

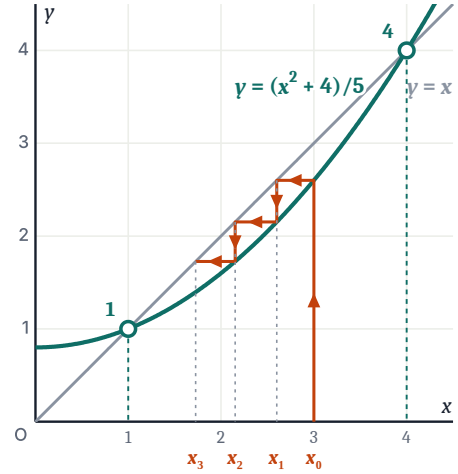
Below, above, below 0.8526, jumps shrinking slowly: a **cobweb** that converges.

either side: cobweb

5 $x_{n+1} = \dfrac{x_n^2 + 4}{5}$, $x_0 = 3$

$x_1 = \dfrac{3^2 + 4}{5} = 2.6000$ substitute x_0

$x_2 = 2.1520$, $x_3 = 1.7262$ ANS key: full value



staircase down to 1

Decreasing: a **staircase** down to the root 1 (it moves away from the other root, 4).

one way: staircase

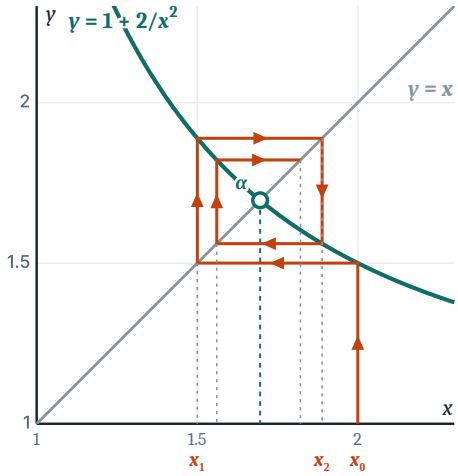
6 (a) Use $x_{n+1} = 1 + \dfrac{2}{x_n^2}$ with $x_0 = 2$ to complete the table (4 d.p.).
(b) Staircase or cobweb? Does it converge?

n	0	1	2	3	4
x_n	2	1.5000	1.8889	1.5606	1.8212

(a) $x_1 = 1 + \dfrac{2}{2^2} = 1.5000$ substitute x_0

Carry on with the ANS key (click the cells to check x_2, x_3, x_4).

ANS key: full value



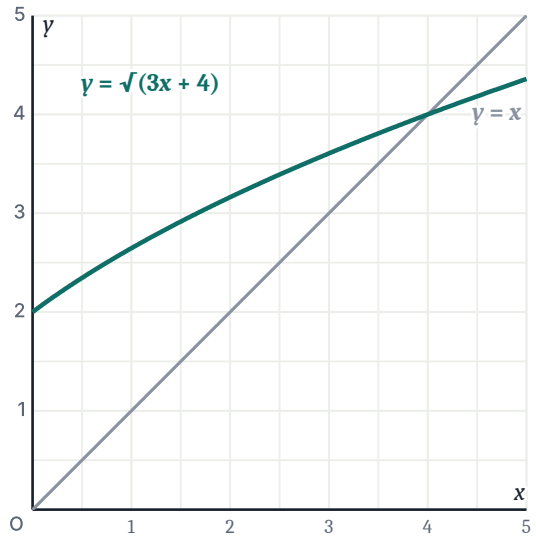
cobweb

(b) Below, above, below, above 1.6956, closer each time (slowly): a **cobweb** that converges.

either side: cobweb

B · Drawing diagrams Draw on the printed graph with a ruler. Mark the iterates on the x-axis and put arrows on the path.

7 Use $x_{n+1} = \sqrt{3x_n + 4}$ with $x_0 = 1$ to find x_1 and x_2. On the graph, draw the diagram and mark x_1 and x_2. Which root does it approach?

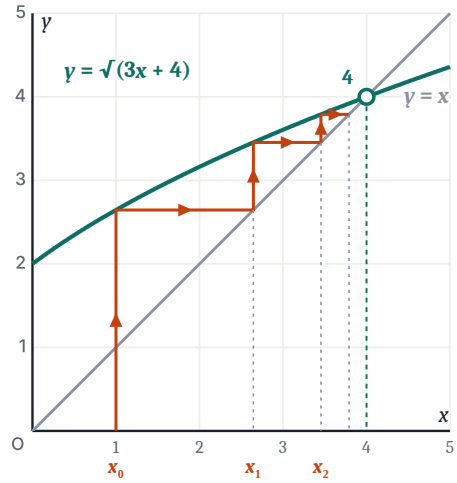


$x_1 = \sqrt{3 \times 1 + 4} = 2.6458$ substitute x_0

$x_2 = 3.4550$ ANS key: full value

Up to the curve, across to $y = x$, mark x_1; up, across, mark x_2.

up, then across

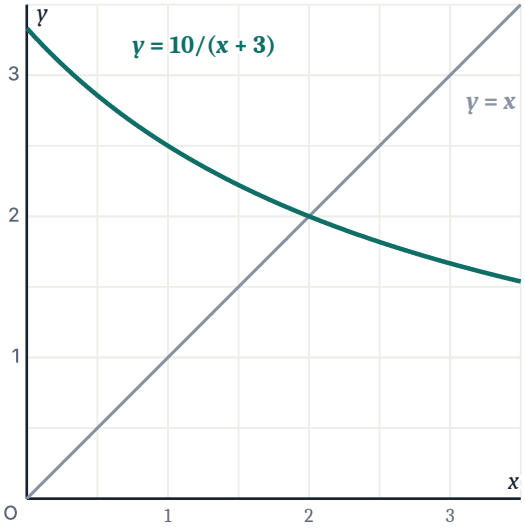


a staircase up to 4

A staircase climbing to $\alpha = 4$: check $\sqrt{3 \times 4 + 4} = 4$.

converges to α

8 Use $x_{n+1} = \dfrac{10}{x_n + 3}$ with $x_0 = 0.5$ to find x_1, x_2, x_3. On the graph, draw the diagram and mark them. Describe it.

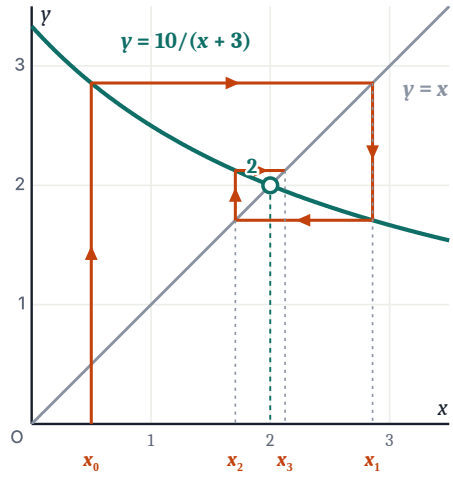


$x_1 = \dfrac{10}{0.5 + 3} = 2.8571$ substitute x_0

$x_2 = 1.7073$, $x_3 = 2.1244$ ANS key: full value

Up to the curve, across to $y = x$; down, across; up, across. Mark x_1, x_2, x_3.

up, down, up

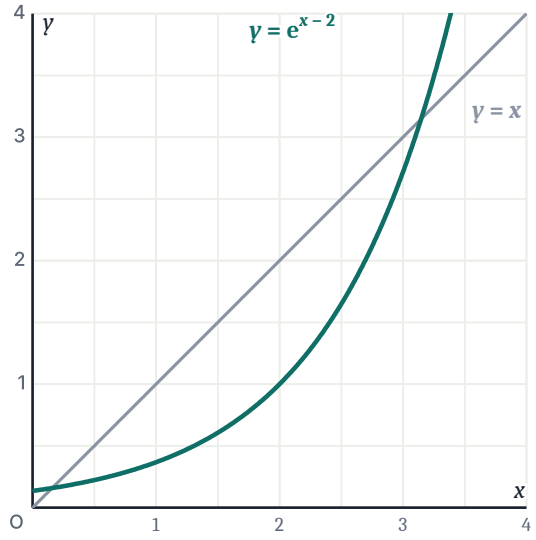


a cobweb closing in on 2

Either side of 2, closer each time: a **cobweb** that converges to $\alpha = 2$.

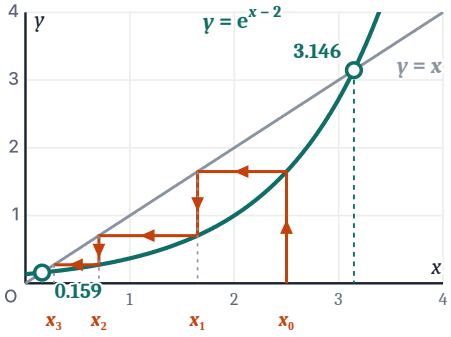
cobweb: converges

9 The graphs of $y = x$ and $y = e^{x-2}$ meet twice.
(a) Use $x_{n+1} = e^{x_n - 2}$ with $x_0 = 2.5$ to find x_1, x_2, x_3. Draw the diagram. Which root does it approach?
(b) Use the diagram to explain what happens from $x_0 = 3.3$.



(a) $x_1 = e^{2.5-2} = 1.6487$ substitute x_0
$x_2 = 0.7038$, $x_3 = 0.2736$ ANS key: full value
Down to the curve, across, down...: a **staircase** down to the smaller root 0.1586.

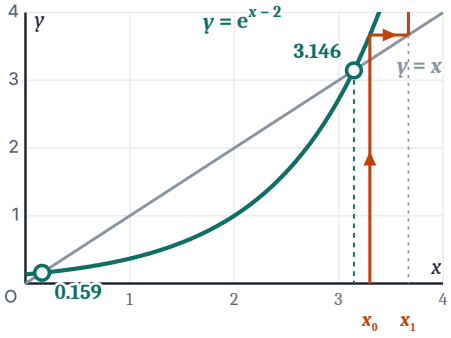
staircase: converges



(a) a staircase down to the smaller root

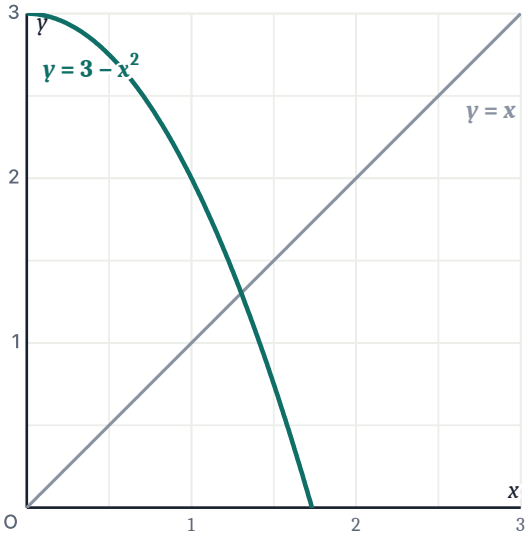
(b) At 3.3 the curve is above $y = x$ and much steeper: $x_1 = 3.6693, x_2 = 5.3084$. The staircase climbs away from the larger root 3.1462: it diverges.

staircase: diverges



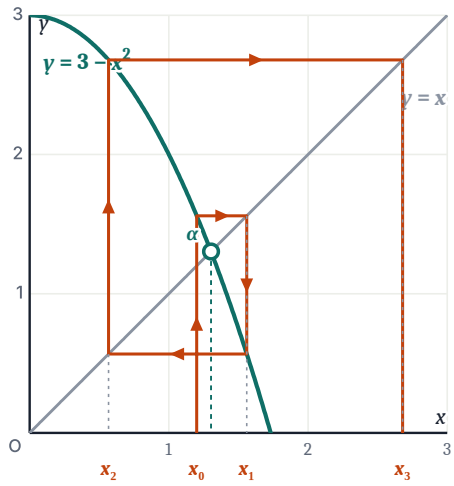
(b) climbs away from the larger root

10 Use $x_{n+1} = 3 - x_n^2$ with $x_0 = 1.2$ to find x_1, x_2, x_3. On the graph, draw the diagram and mark them. Does it converge?



$x_1 = 3 - (1.2)^2 = 1.5600$ substitute x_0
$x_2 = 0.5664$, $x_3 = 2.6792$ ANS key: full value
Up, across, down, across, up, across: round $\alpha = 1.3028$ but further out each time.

cobweb: diverges



a cobweb spiralling out

No: a **cobweb that diverges** (it spirals out). Next, $x_4 = -4.1781$.

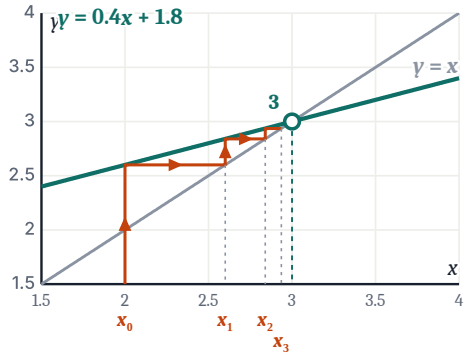
spirals out

C · Reasoning Explain fully. Q11: no calculator.

11 Two iterations start at $x_0 = 2$:
(i) 2, 2.6, 2.84, 2.936, ...
(ii) 2, 3.5, 2.75, 3.125, ...
(a) Would each diagram be a staircase or a cobweb? What value does each approach?
(b) Sequence (i) comes from $x_{n+1} = 0.4x_n + 1.8$. Find its limit exactly.

(a) (i) always increasing, gaps 0.6, 0.24, 0.096: a **staircase** approaching 3.

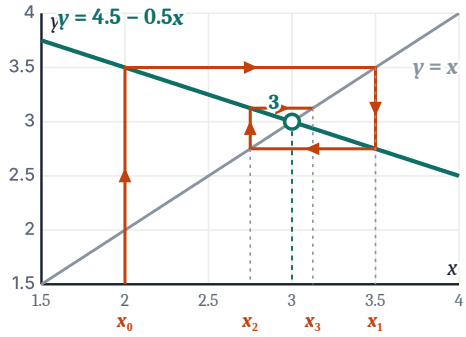
one way: staircase



(i) a staircase

(ii) above, below, above 3, gaps 1.5, 0.75, 0.375: a **cobweb** approaching 3.

either side: cobweb



(ii) a cobweb

(b) $\alpha = 0.4\alpha + 1.8 \Rightarrow 0.6\alpha = 1.8 \Rightarrow \alpha = 3$ limit: α = g(α)

12 $x^2 - 4x - 1 = 0$ has roots $\alpha < 0 < \beta$.
(a) Show that it can be written as $x = \dfrac{x^2 - 1}{4}$.
(b) With $x_0 = 0$, find x_1, x_2, x_3. Describe the diagram.
(c) Use a sketch of $y = x$ and $y = \dfrac{x^2 - 1}{4}$ to explain why this formula cannot find β, even from $x_0 = 4$.

(a) $4x = x^2 - 1 \Rightarrow x = \dfrac{x^2 - 1}{4}$ 4x to one side, ÷ 4

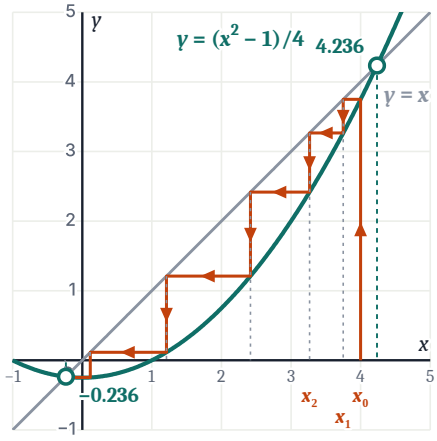
(b) $x_1 = \dfrac{0^2 - 1}{4} = -0.2500$ substitute x_0

$x_2 = -0.2344$, $x_3 = -0.2363$ ANS key: full value
Either side of $\alpha = 2 - \sqrt{5} = -0.2361$, closer each time: a cobweb that converges to α.

cobweb: converges

(c) Near $\beta = 2 + \sqrt{5} = 4.2361$ the parabola is steeper than $y = x$. From 4 the curve is under the line, so the path steps **down**, away from β (3.7500, 3.2656, ...) and ends at α. Above β it climbs away.

explain from the diagram



from 4: away from β, to α

13 *Extension (not on the specification).* $x^2 - 2x - 1 = 0$ has the root $\alpha = 1 + \sqrt{2}$ (Explore).

(a) For $g(x) = \sqrt{2x + 1}$, find $g'(x)$ and show that $g'(\alpha) = \dfrac{1}{\alpha}$.

(b) For $h(x) = \dfrac{x^2 - 1}{2}$, find $h'(\alpha)$.

(c) Use the rule "converges if $-1 < g'(\alpha) < 1$" to explain the two Explore diagrams.

(a) $g'(x) = \dfrac{1}{\sqrt{2x + 1}}$ — differentiate

At α: $\sqrt{2\alpha + 1} = \alpha$ (because $\alpha = g(\alpha)$), so

$g'(\alpha) = \dfrac{1}{\alpha} = \dfrac{1}{1 + \sqrt{2}} = \sqrt{2} - 1 = 0.4142$

α = g(α)

(b) $h'(x) = x$, so $h'(\alpha) = \alpha = 2.4142$ — differentiate

(c) $0 < g'(\alpha) < 1$: converges, and positive, so a staircase. $h'(\alpha) > 1$: diverges, a staircase moving away (Explore Q1 and Q3).

compare with 1

D · Show that

14 $f(x) = x^3 - 5x - 3$.

(a) Show that $f(x) = 0$ has a root α in $[2.4, 2.5]$.

(b) Show that $f(x) = 0$ can be written as $x = \sqrt[3]{5x + 3}$.

(c) Use $x_{n+1} = \sqrt[3]{5x_n + 3}$ with $x_0 = 1$ to find x_1, x_2, x_3.

(d) Staircase or cobweb? Explain.

(e) Show that $\alpha = 2.491$ correct to 3 d.p.

(a) $f(2.4) = -1.176 < 0$, $f(2.5) = 0.125 > 0$: change of sign, f continuous, so a root in $[2.4, 2.5]$.

continuous ⇒ root

(b) $x^3 = 5x + 3 \Rightarrow x = \sqrt[3]{5x + 3}$

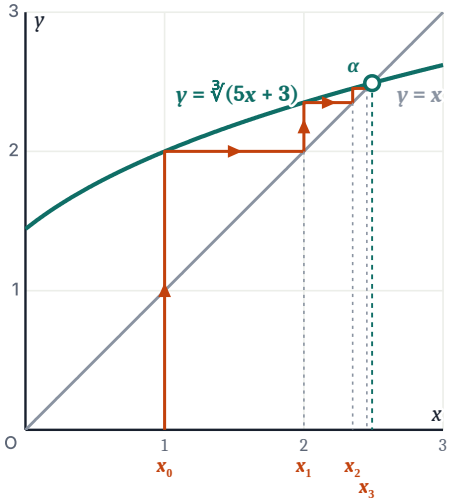
cube root

(c) $x_1 = \sqrt[3]{5 \times 1 + 3} = 2.0000$, $x_2 = 2.3513$, $x_3 = 2.4528$

substitute x_0, ANS key

(d) Increasing: a **staircase**.

one way: staircase



(d) a staircase

(e) Bounds 2.4905 and 2.4915: $f(2.4905) = -0.00495 < 0$, $f(2.4915) = 0.00867 > 0$

bounds: ±0.0005

Change of sign and f is continuous, so $2.4905 < \alpha < 2.4915$. So $\alpha = 2.491$ to 3 d.p.

continuous ⇒ root

15 $f(x) = e^x - x - 3$ has roots $\beta < 0 < \alpha$.

(a) Show that $f(x) = 0$ can be written as $x = \ln(x + 3)$.

(b) Use $x_{n+1} = \ln(x_n + 3)$ with $x_0 = 1$ to find x_1, x_2, x_3. Describe the diagram.

(c) Show that $f(x) = 0$ can also be written as $x = e^x - 3$. Use this with $x_0 = 0$ to find x_1, x_2, x_3. Which root does it approach?

(d) Explain why $x_{n+1} = e^{x_n} - 3$ cannot be used to find α.

(a) $e^x = x + 3 \Rightarrow x = \ln(x + 3)$

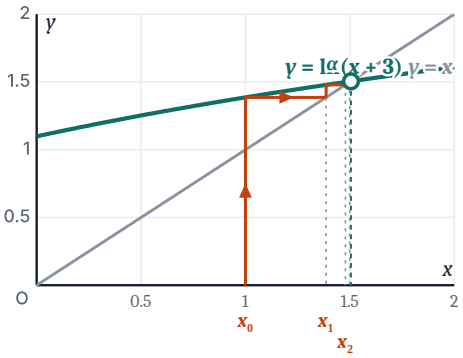
take ln of both sides

(b) $x_1 = \ln(1 + 3) = 1.3863$, $x_2 = 1.4785$, $x_3 = 1.4993$

substitute x_0, ANS key

Increasing: a staircase up to $\alpha = 1.5052$.

staircase: converges



(b) a staircase

(c) $e^x - 3 = x \Rightarrow x = e^x - 3$

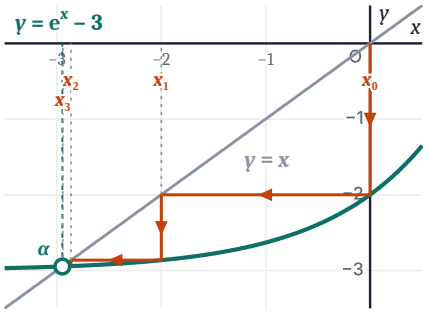
3 to the other side

$x_1 = e^0 - 3 = -2.0000$, $x_2 = -2.8647$, $x_3 = -2.9430$

substitute x_0, ANS key

Decreasing: a staircase down to $\beta = -2.9475$.

staircase: converges



(c) a staircase down to β

(d) Near α, $y = e^x - 3$ is much steeper than $y = x$ (gradient $e^\alpha \approx 4.5$): the staircase moves away from α.

explain from the diagram

16 $f(x) = x - \cos x$ (x in radians).

(a) Show that $f(x) = 0$ has a root α in $[0.7, 0.8]$.

(b) Use $x_{n+1} = \cos x_n$ with $x_0 = 1$ to find x_1, x_2, x_3, x_4.

(c) Staircase or cobweb? Explain.

(d) Show that $\alpha = 0.739$ correct to 3 d.p.

(a) $f(0.7) = -0.0648 < 0$, $f(0.8) = 0.103 > 0$: change of sign, f continuous, so a root in $[0.7, 0.8]$.

continuous ⇒ root

(b) $x_1 = \cos 1 = 0.5403$, $x_2 = 0.8576$, $x_3 = 0.6543$, $x_4 = 0.7935$

substitute, ANS key (radians)

(c) Either side of α, closer each time: a **cobweb**.

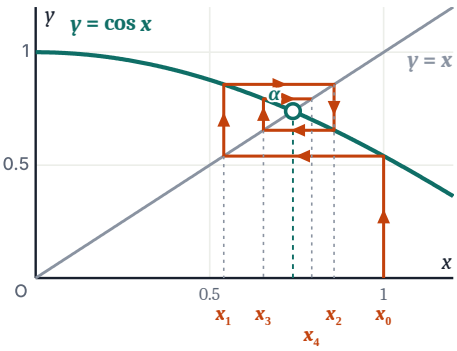
either side: cobweb

(d) Bounds 0.7385 and 0.7395: $f(0.7385) = -0.000979 < 0$, $f(0.7395) = 0.000694 > 0$

bounds: ±0.0005

Change of sign and f is continuous, so $0.7385 < \alpha < 0.7395$. So $\alpha = 0.739$ to 3 d.p.

continuous ⇒ root



(c) a cobweb

E · Exam-style question AQA-style (AQA starts from x_1). Use the mark scheme to self-assess.

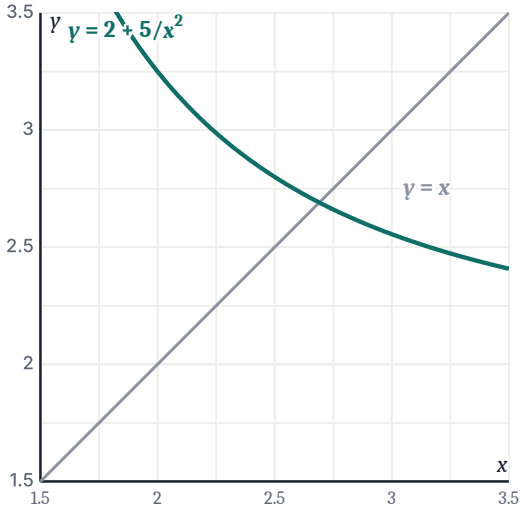
17 $f(x) = x^3 - 2x^2 - 5$ has a single root α.

(a) Show that α lies between 2.6 and 2.7. [2 marks]

(b) Show that $f(x) = 0$ can be rearranged into the form $x = 2 + \frac{5}{x^2}$. [2 marks]

(c) Use $x_{n+1} = 2 + \frac{5}{x_n^2}$ with $x_1 = 2$ to find x_2 and x_3, giving your answers to three decimal places. [2 marks]

(d) On the figure (graphs of $y = x$ and $y = 2 + \frac{5}{x^2}$), draw a cobweb or staircase diagram to show how convergence takes place, indicating the positions of x_2 and x_3 on the x-axis. [2 marks]



(a) $f(2.6) = -0.944 < 0$, $f(2.7) = 0.103 > 0$ **M1** substitute both ends

Change of sign and f is continuous, so $2.6 < \alpha < 2.7$. **A1**

continuous ⇒ root

(b) $x^3 - 2x^2 = 5 \Rightarrow x^2(x-2) = 5 \Rightarrow x - 2 = \frac{5}{x^2}$ **M1** factorise x^2, ÷ x^2

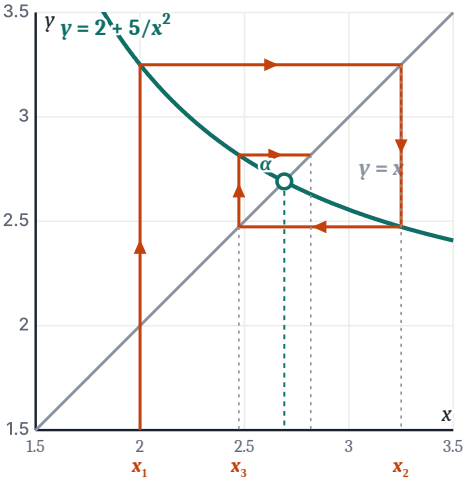
$x = 2 + \frac{5}{x^2}$ $(x \neq 0)$ **A1** + 2

(c) $x_2 = 2 + \frac{5}{2^2} = 3.25$, $x_3 = 2 + \frac{5}{3.25^2} = 2.473$ **B1 B1**

substitute x_1, then x_2

(d) Cobweb: up from $x_1 = 2$ to the curve, across to $y = x$, down, across; x_2 and x_3 marked either side of α. **M1 A1**

up, across, down, across



a cobweb