Staircase and Cobweb Diagrams

Specification: AQA 7357: I2 · Edexcel 9MA0: 9.2

INDEPENDENT

You do

A · Staircase or cobweb? Find x_1, x_2 and x_3 to 4 decimal places. Would the diagram be a staircase or a cobweb? Does it converge?

1 $x_{n+1} = \sqrt{x_n + 5}$, $x_0 = 1$

2 $x_{n+1} = \dfrac{4}{x_n + 3}$, $x_0 = 0$

3 $x_{n+1} = 3 + \ln x_n$, $x_0 = 3$

4 $x_{n+1} = 2e^{-x_n}$, $x_0 = 1$

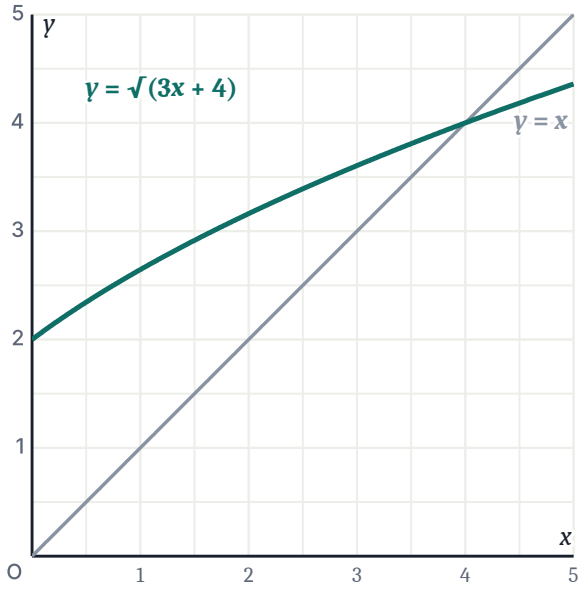
5 $x_{n+1} = \dfrac{x_n^{\,2} + 4}{5}$, $x_0 = 3$

6 (a) Use $x_{n+1} = 1 + \dfrac{2}{x_n^{\,2}}$ with $x_0 = 2$ to complete the table (4 d.p.).
(b) Staircase or cobweb? Does it converge?

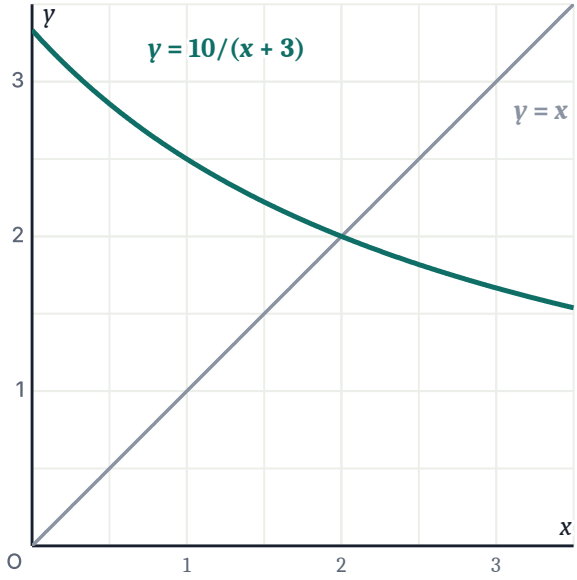
n	0	1	2	3	4
x_n	2				

B · Drawing diagrams Draw on the printed graph with a ruler. Mark the iterates on the x-axis and put arrows on the path.

7 Use $x_{n+1} = \sqrt{3x_n + 4}$ with $x_0 = 1$ to find x_1 and x_2. On the graph, draw the diagram and mark x_1 and x_2. Which root does it approach?



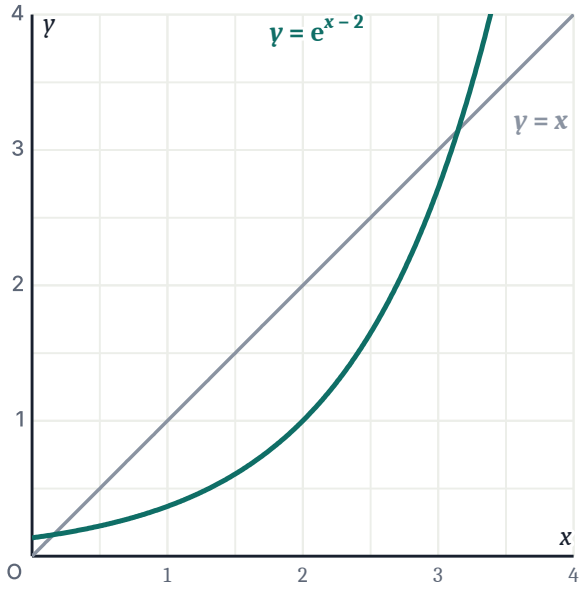
8 Use $x_{n+1} = \dfrac{10}{x_n + 3}$ with $x_0 = 0.5$ to find x_1, x_2, x_3. On the graph, draw the diagram and mark them. Describe it.



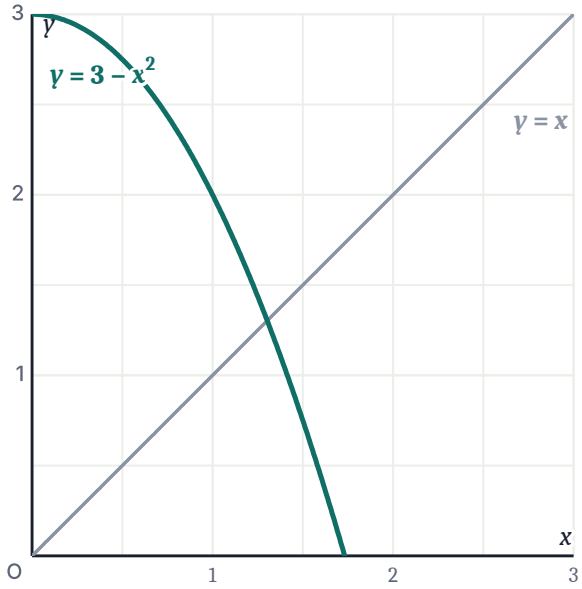
9 The graphs of $y = x$ and $y = e^{x-2}$ meet twice.

(a) Use $x_{n+1} = e^{x_n - 2}$ with $x_0 = 2.5$ to find x_1, x_2, x_3. Draw the diagram. Which root does it approach?

(b) Use the diagram to explain what happens from $x_0 = 3.3$.



10 Use $x_{n+1} = 3 - x_n^2$ with $x_0 = 1.2$ to find x_1, x_2, x_3. On the graph, draw the diagram and mark them. Does it converge?



C · Reasoning Explain fully. Q11: no calculator.

11 Two iterations start at $x_0 = 2$:
(i) 2, 2.6, 2.84, 2.936, ...
(ii) 2, 3.5, 2.75, 3.125, ...
(a) Would each diagram be a staircase or a cobweb? What value does each approach?
(b) Sequence (i) comes from $x_{n+1} = 0.4x_n + 1.8$. Find its limit exactly.

12 $x^2 - 4x - 1 = 0$ has roots $\alpha < 0 < \beta$.
(a) Show that it can be written as $x = \frac{x^2 - 1}{4}$.
(b) With $x_0 = 0$, find x_1, x_2, x_3. Describe the diagram.
(c) Use a sketch of $y = x$ and $y = \frac{x^2 - 1}{4}$ to explain why this formula cannot find β, even from $x_0 = 4$.

13 *Extension (not on the specification).* $x^2 - 2x - 1 = 0$ has the root $\alpha = 1 + \sqrt{2}$ (Explore).
(a) For $g(x) = \sqrt{2x + 1}$, find $g'(x)$ and show that $g'(\alpha) = \frac{1}{\alpha}$.
(b) For $h(x) = \frac{x^2 - 1}{2}$, find $h'(\alpha)$.
(c) Use the rule "converges if $-1 < g'(\alpha) < 1$" to explain the two Explore diagrams.

D · Show that

14 $f(x) = x^3 - 5x - 3$.
(a) Show that $f(x) = 0$ has a root α in $[2.4, 2.5]$.
(b) Show that $f(x) = 0$ can be written as $x = \sqrt[3]{5x + 3}$.
(c) Use $x_{n+1} = \sqrt[3]{5x_n + 3}$ with $x_0 = 1$ to find x_1, x_2, x_3.
(d) Staircase or cobweb? Explain.
(e) Show that $\alpha = 2.491$ correct to 3 d.p.

15 $f(x) = e^x - x - 3$ has roots $\beta < 0 < \alpha$.
(a) Show that $f(x) = 0$ can be written as $x = \ln(x + 3)$.
(b) Use $x_{n+1} = \ln(x_n + 3)$ with $x_0 = 1$ to find x_1, x_2, x_3. Describe the diagram.
(c) Show that $f(x) = 0$ can also be written as $x = e^x - 3$. Use this with $x_0 = 0$ to find x_1, x_2, x_3. Which root does it approach?
(d) Explain why $x_{n+1} = e^{x_n} - 3$ cannot be used to find α.

16 $f(x) = x - \cos x$ (x in radians).

(a) Show that $f(x) = 0$ has a root α in $[0.7, 0.8]$.

(b) Use $x_{n+1} = \cos x_n$ with $x_0 = 1$ to find x_1, x_2, x_3, x_4.

(c) Staircase or cobweb? Explain.

(d) Show that $\alpha = 0.739$ correct to 3 d.p.

E · Exam-style question AQA-style (AQA starts from x_1). Use the mark scheme to self-assess.

17 $f(x) = x^3 - 2x^2 - 5$ has a single root α.

(a) Show that α lies between 2.6 and 2.7. [2 marks]

(b) Show that $f(x) = 0$ can be rearranged into the form $x = 2 + \dfrac{5}{x^2}$. [2 marks]

(c) Use $x_{n+1} = 2 + \dfrac{5}{x_n^{\,2}}$ with $x_1 = 2$ to find x_2 and x_3, giving your answers to three decimal places. [2 marks]

(d) On the figure (graphs of $y = x$ and $y = 2 + \dfrac{5}{x^2}$), draw a cobweb or staircase diagram to show how convergence takes place, indicating the positions of x_2 and x_3 on the x-axis. [2 marks]

