The Newton–Raphson Method

Specification: AQA 7357: I2 · Edexcel 9MA0: 9.3

INDEPENDENT **You do**

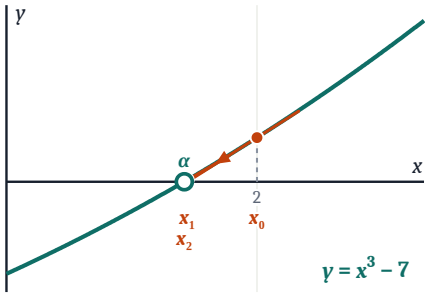
A · Applying Newton–Raphson Find $f'(x)$, then use x_0 to find x_1 and x_2 to 4 decimal places.

1 $f(x) = x^3 - 7$, $x_0 = 2$

$f'(x) = 3x^2$ — differentiate

$x_1 = 2 - \frac{1}{12} = 1.9167$ — N–R formula: x_0

$x_2 = 1.9167\ldots - \frac{0.04109\ldots}{11.02\ldots} = 1.9129$ — substitute x_1: full value



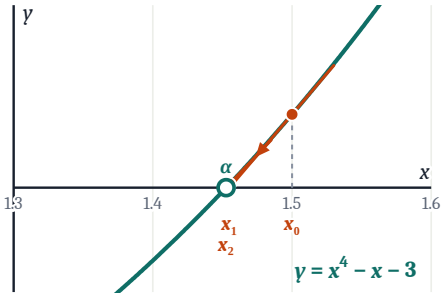
tangents to $\sqrt[3]{7}$

2 $f(x) = x^4 - x - 3$, $x_0 = 1.5$

$f'(x) = 4x^3 - 1$ — differentiate

$x_1 = 1.5 - \frac{0.5625}{12.5} = 1.455$ — N–R formula: x_0

$x_2 = 1.455 - \frac{0.02679\ldots}{11.32\ldots} = 1.4526$ — substitute x_1: full value



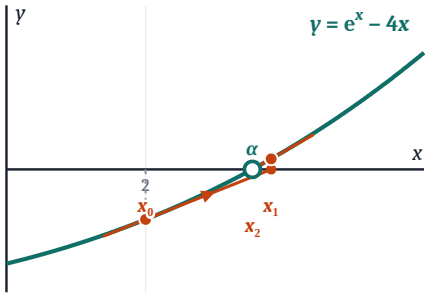
tangents to α

3 $f(x) = e^x - 4x$, $x_0 = 2$ (Lesson 2)

$f'(x) = e^x - 4$ — differentiate

$x_1 = 2 - \frac{(-0.6109\ldots)}{3.389\ldots} = 2.1803$ — N–R formula: x_0

$x_2 = 2.1803\ldots - \frac{0.1276\ldots}{4.849\ldots} = 2.1540$ — substitute x_1: full value



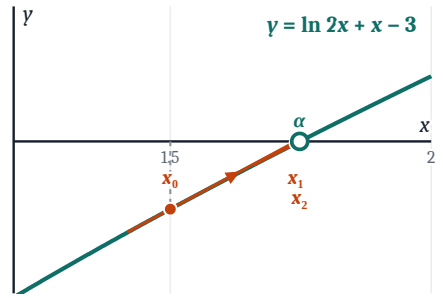
tangents to α

4 $f(x) = \ln 2x + x - 3$, $x > 0$, $x_0 = 1.5$

$f'(x) = \frac{2}{2x} + 1 = \frac{1}{x} + 1$ — differentiate: chain rule

$x_1 = 1.5 - \frac{(-0.4014\ldots)}{1.667\ldots} = 1.7408$ — N–R formula: x_0

$x_2 = 1.7408\ldots - \frac{(-0.01166\ldots)}{1.574\ldots} = 1.7482$ — substitute x_1: full value



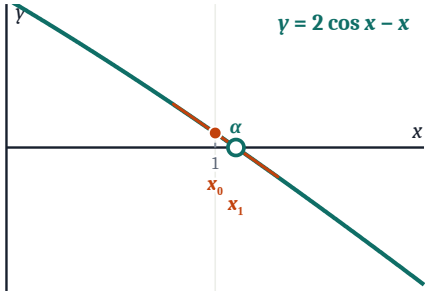
tangents to α

5 $f(x) = 2\cos x - x$ (x in radians), $x_0 = 1$

$f'(x) = -2\sin x - 1$ — differentiate

$x_1 = 1 - \dfrac{0.08060\ldots}{-2.683\ldots} = 1.0300$ — N–R formula: x_0

$x_2 = 1.0300\ldots - \dfrac{(-0.0004800\ldots)}{-2.715\ldots} = 1.0299$ — substitute x_1 (radians)



tangents to α

6 (a) Use the Newton–Raphson method on $f(x) = x^3 + 3x^2 - 2$ with $x_0 = 1$ to complete the table (4 d.p.).
(b) Show that $f(x) = (x+1)(x^2 + 2x - 2)$ and find the exact value the iterates approach.

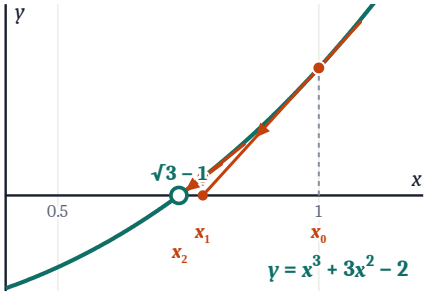
n	0	1	2	3	4
x_n	1	0.7778	0.7338	0.7321	0.7321

(a) $f'(x) = 3x^2 + 6x$ — differentiate

$x_1 = 1 - \dfrac{2}{9} = 0.7778$ — N–R formula: x_0

Carry on with the ANS key (click the cells to check x_2, x_3, x_4).

ANS key: full value



(a) tangents to √3 − 1

(b) $(x+1)(x^2 + 2x - 2)$
$= x^3 + 2x^2 - 2x + x^2 + 2x - 2 = x^3 + 3x^2 - 2$ ✓

expand the brackets

$x^2 + 2x - 2 = 0 \Rightarrow x = -1 \pm \sqrt{3}$. The iterates approach $\sqrt{3} - 1 = 0.7321$

quadratic formula

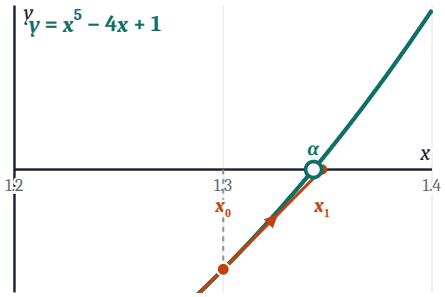
B · Stating the recurrence Write down the Newton–Raphson recurrence, then use it to find x_1 to 4 decimal places.

7 $f(x) = x^5 - 4x + 1$, $x_0 = 1.3$

$f'(x) = 5x^4 - 4$ — differentiate

$x_{n+1} = x_n - \dfrac{x_n^5 - 4x_n + 1}{5x_n^4 - 4}$ — N–R formula

$x_1 = 1.3 - \dfrac{(-0.48707)}{10.2805} = 1.3474$ — N–R formula: x_0



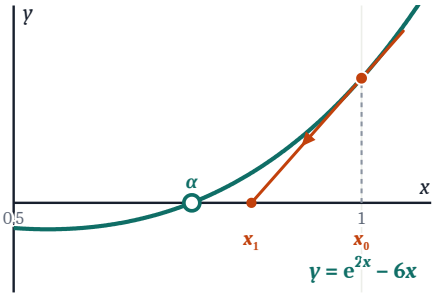
one tangent: x_1

8 $f(x) = e^{2x} - 6x$, $x_0 = 1$

$f'(x) = 2e^{2x} - 6$ — differentiate: chain rule

$x_{n+1} = x_n - \dfrac{e^{2x_n} - 6x_n}{2e^{2x_n} - 6}$ — N–R formula

$x_1 = 1 - \dfrac{1.389\ldots}{8.778\ldots} = 0.8418$ — N–R formula: x_0



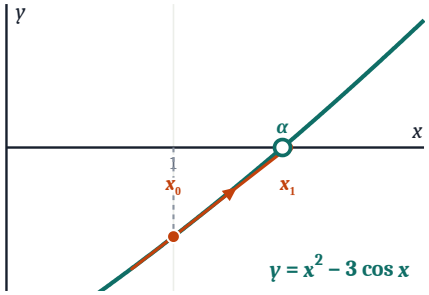
one tangent: x_1

9 $f(x) = x^2 - 3\cos x$ (x in radians), $x_0 = 1$

$f'(x) = 2x + 3\sin x$ — differentiate

$x_{n+1} = x_n - \dfrac{x_n^2 - 3\cos x_n}{2x_n + 3\sin x_n}$ — N–R formula

$x_1 = 1 - \dfrac{(-0.6209\ldots)}{4.524\ldots} = 1.1372$ — N–R formula: x_0



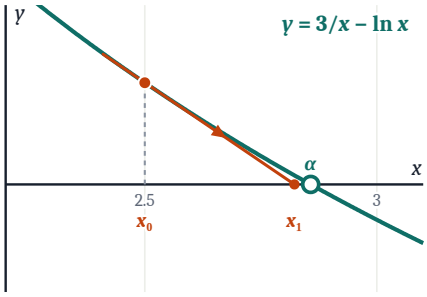
one tangent: x_1

10 $f(x) = \dfrac{3}{x} - \ln x$, $x > 0$, $x_0 = 2.5$

$f'(x) = -\dfrac{3}{x^2} - \dfrac{1}{x}$ — differentiate

$x_{n+1} = x_n - \dfrac{\frac{3}{x_n} - \ln x_n}{-\frac{3}{x_n^2} - \frac{1}{x_n}}$ — N–R formula

$x_1 = 2.5 - \dfrac{0.2837\ldots}{-0.88} = 2.8224$ — N–R formula: x_0



one tangent: x_1

C · Reasoning Explain fully. Q11 and Q13: no calculator.

11 (a) Show that the Newton–Raphson recurrence for $f(x) = x^2 - 10$ can be written as $x_{n+1} = \dfrac{1}{2}\left(x_n + \dfrac{10}{x_n}\right)$.

(b) Using $x_0 = 3$, find x_1 and x_2 as fractions.

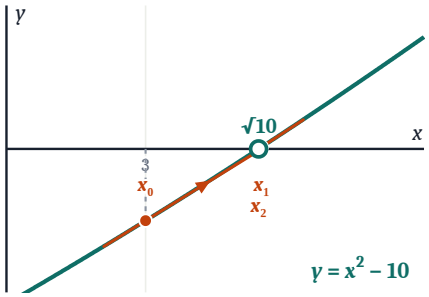
(c) How close is x_2 to $\sqrt{10}$?

(a) $x_{n+1} = x_n - \dfrac{x_n^2 - 10}{2x_n} = \dfrac{2x_n^2 - x_n^2 + 10}{2x_n} = \dfrac{x_n^2 + 10}{2x_n}$ — common denominator

$= \dfrac{1}{2}\left(x_n + \dfrac{10}{x_n}\right)$ ✓ — split the fraction

(b) $x_1 = \dfrac{1}{2}\left(3 + \dfrac{10}{3}\right) = \dfrac{19}{6}$ — substitute x_0

$x_2 = \dfrac{1}{2}\left(\dfrac{19}{6} + \dfrac{60}{19}\right) = \dfrac{1}{2} \times \dfrac{361 + 360}{114} = \dfrac{721}{228}$ — substitute x_1



(b) tangents to √10

(c) $\left(\dfrac{721}{228}\right)^2 = \dfrac{519841}{51984}$, only $\dfrac{1}{51984}$ more than 10: x_2 agrees with $\sqrt{10} = 3.16228$ to 5 d.p.

compare with 10

12 $f(x) = \dfrac{x}{\sqrt{1+x^2}}$ has the root 0 (Explore Q4).

(a) Show that $f'(x) = \dfrac{1}{(1+x^2)^{3/2}}$.

(b) Hence show that the Newton–Raphson recurrence is $x_{n+1} = -x_n^3$.

(c) Describe what happens from $x_0 = 0.5$, from $x_0 = 1$ and from $x_0 = 1.2$.

(a)

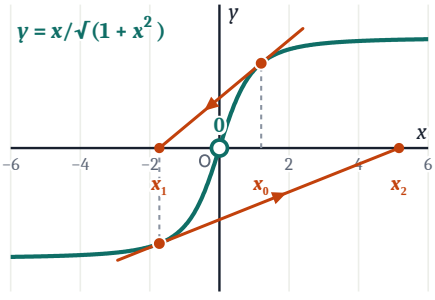
$f'(x) = \dfrac{\sqrt{1+x^2} - x \times \frac{x}{\sqrt{1+x^2}}}{1+x^2} = \dfrac{(1+x^2) - x^2}{(1+x^2)^{3/2}} = \dfrac{1}{(1+x^2)^{3/2}}$ — quotient rule

(b)

$x_{n+1} = x_n - \dfrac{x_n}{\sqrt{1+x_n^2}} \times (1+x_n^2)^{3/2} = x_n - x_n(1+x_n^2) = -x_n^3$ — divide: invert and multiply

(c) From 0.5: −0.125, 0.0020, … converges to 0. From 1: −1, 1, −1, … oscillates. — cube each time

From 1.2: −1.728, 5.1598, … either side of 0, further out: diverges. It converges only if $|x_0| < 1$. — alternates sign: diverges



from 1.2: either side, further out

13 $f(x) = (x - 2)^2$ has the root 2.

(a) Explain why a change of sign cannot be used to locate this root.

(b) Show that the Newton–Raphson recurrence is $x_{n+1} = \dfrac{x_n + 2}{2}$.

(c) Using $x_0 = 3$, find x_1, x_2, x_3. How quickly do they approach 2?

(a) $f(x) \geq 0$ for every x: the curve touches the x-axis at 2 and never goes below it. No change of sign (Lesson 1).

repeated root: no sign change

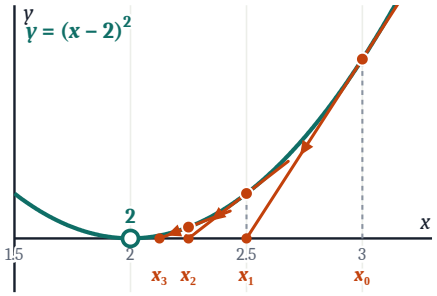
(b) $x_{n+1} = x_n - \dfrac{(x_n - 2)^2}{2(x_n - 2)} = x_n - \dfrac{x_n - 2}{2} = \dfrac{x_n + 2}{2}$

cancel $(x_n - 2)$

(c) $x_1 = 2.5, x_2 = 2.25, x_3 = 2.125$ substitute each value

The distance to 2 halves each time (1, 0.5, 0.25, 0.125): it converges, but slowly for N–R.

halves each time



tangents halve the distance to 2

14 $f(x) = x^3 - 12x + 5$ has three roots, near −3.66, 0.42 and 3.23.

(a) Find the x-coordinates of the stationary points.

(b) Using $x_0 = 1.9$, find x_1. Explain why it is far from x_0.

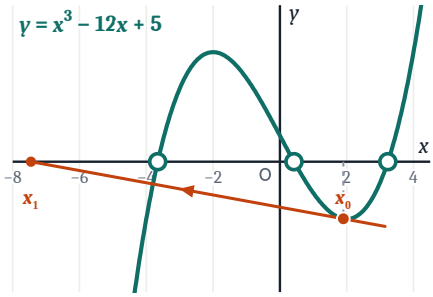
(c) From $x_0 = 1.9$ the iterates go to −3.656. Choose a better x_0 for the root near 3.23, and find x_1.

(a) $f'(x) = 3x^2 - 12 = 0 \Rightarrow x = \pm 2$ stationary: f′(x) = 0

(b) $x_1 = 1.9 - \dfrac{(-10.941)}{-1.17} = -7.4513$ N–R formula: x_0

$f'(1.9) = -1.17$ is small (1.9 is near the stationary point at 2): the tangent is nearly flat and meets the x-axis far to the left.

f′ small: far away



nearly flat tangent at 1.9: $x_1 = -7.45$

(c) Start beyond the stationary point, e.g. $x_0 = 3.5$:
$x_1 = 3.5 - \dfrac{5.875}{24.75} = 3.2626$, heading for 3.2332.

start away from x = 2

D · Show that

15 $f(x) = x^3 + 2x^2 - 4$.

(a) Show that $f(x) = 0$ has a root α in $[1.1, 1.2]$.

(b) Using $x_0 = 1$, find x_1 and x_2 by Newton–Raphson (4 d.p.).

(c) Show that $\alpha = 1.130$ to 3 d.p.

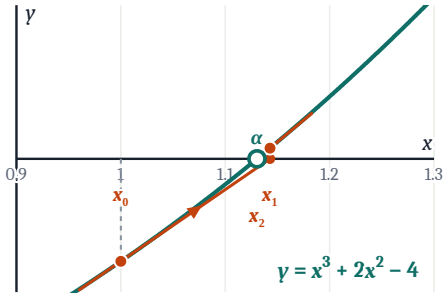
(a) $f(1.1) = -0.249 < 0$, $f(1.2) = 0.608 > 0$: change of sign, f continuous ⇒ a root in $[1.1, 1.2]$.

continuous ⇒ root

(b) $f'(x) = 3x^2 + 4x$ — differentiate

$x_1 = 1 - \dfrac{(-1)}{7} = 1.1429$ — N–R formula: x_0

$x_2 = 1.1429\ldots - \dfrac{0.1050\ldots}{8.490\ldots} = 1.1305$ — substitute x_1: full value



(b) tangents to α

(c) $f(1.1295) = -0.00748 < 0$, $f(1.1305) = 0.000874 > 0$

bounds: ±0.0005

Change of sign, f continuous ⇒ $1.1295 < \alpha < 1.1305$, so $\alpha = 1.130$ (3 d.p.)

continuous ⇒ root

16 $f(x) = e^x - 2x - 2$.

(a) Show that $f(x) = 0$ has a root α in $[1.6, 1.7]$.

(b) Using $x_0 = 2$, find x_1 and x_2 by Newton–Raphson (4 d.p.).

(c) Show that $\alpha = 1.678$ to 3 d.p.

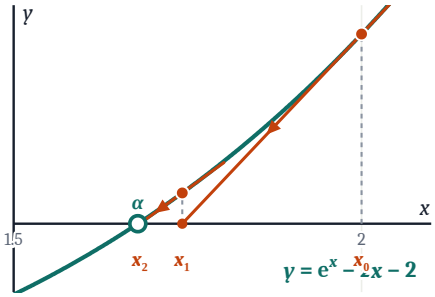
(a) $f(1.6) = -0.247 < 0$, $f(1.7) = 0.0739 > 0$: change of sign, f continuous ⇒ a root in $[1.6, 1.7]$.

continuous ⇒ root

(b) $f'(x) = e^x - 2$ — differentiate

$x_1 = 2 - \dfrac{1.389\ldots}{5.389\ldots} = 1.7422$ — N–R formula: x_0

$x_2 = 1.7422\ldots - \dfrac{0.2257\ldots}{3.710\ldots} = 1.6814$ — substitute x_1: full value



(b) tangents to α

(c) $f(1.6775) = -0.00284 < 0$, $f(1.6785) = 0.000514 > 0$

bounds: ±0.0005

Change of sign, f continuous ⇒ $1.6775 < \alpha < 1.6785$, so $\alpha = 1.678$ (3 d.p.)

continuous ⇒ root

17 $f(x) = x^2 - 2\sin x - 1$ (x in radians).

(a) Show that $f(x) = 0$ has a root α in $[1.7, 1.8]$.

(b) Using $x_0 = 1.5$, find x_1 and x_2 by Newton–Raphson (4 d.p.).

(c) Show that $\alpha = 1.725$ to 3 d.p.

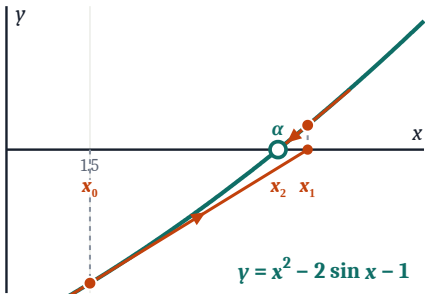
(a) $f(1.7) = -0.0933 < 0$, $f(1.8) = 0.292 > 0$: change of sign, f continuous ⇒ a root in $[1.7, 1.8]$.

continuous ⇒ root

(b) $f'(x) = 2x - 2\cos x$ differentiate

$$x_1 = 1.5 - \frac{(-0.7450\ldots)}{2.859\ldots} = 1.7606$$ N–R formula: x_0

$$x_2 = 1.7606\ldots - \frac{0.1357\ldots}{3.899\ldots} = 1.7258$$ substitute x_1 (radians)



(b) tangents to α

(c) $f(1.7245) = -0.00252 < 0$, $f(1.7255) = 0.00124 > 0$

bounds: ±0.0005

Change of sign, f continuous ⇒ $1.7245 < \alpha < 1.7255$, so $\alpha = 1.725$ (3 d.p.)

continuous ⇒ root

E · Exam-style question AQA-style (AQA starts from x_1). Use the mark scheme to self-assess.

18 $f(x) = x^3 - 3x^2 + 1$. The equation $f(x) = 0$ has a root α between 2 and 3.

(a) Explain why the Newton–Raphson method fails if $x_1 = 2$ is used as a first approximation to α. [2 marks]

(b) Taking $x_1 = 3$, use the Newton–Raphson method to find x_2 and x_3, giving x_3 to three decimal places. [3 marks]

(c) Show that $\alpha = 2.879$ to three decimal places. [3 marks]

(a) $f'(x) = 3x^2 - 6x$ **M1** differentiate

$f'(2) = 12 - 12 = 0$: the tangent at $x = 2$ is parallel to the x-axis (division by 0). **A1**

f'(x₁) = 0: fails

(b) $x_2 = 3 - \frac{1}{9} = \frac{26}{9} = 2.8889$ **M1 A1** N–R formula: x_1

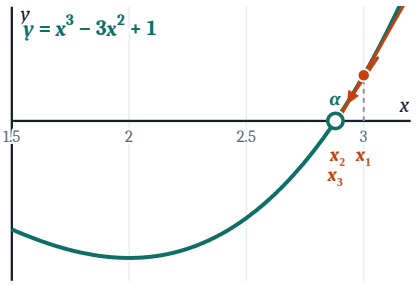
$x_3 = 2.879$ **A1** ANS key: full value

(c) Bounds 2.8785 and 2.8795: $f(2.8785) = -0.00672 < 0$, $f(2.8795) = 0.000872 > 0$ **M1 A1**

substitute both bounds

Change of sign and f is continuous, so $2.8785 < \alpha < 2.8795$: $\alpha = 2.879$ to 3 d.p. **A1**

continuous ⇒ root



(b) tangents from $x_1 = 3$