The Newton–Raphson Method

Specification: AQA 7357: I2 · Edexcel 9MA0: 9.3

INDEPENDENT

You do

A · Applying Newton–Raphson Find $f'(x)$, then use x_0 to find x_1 and x_2 to 4 decimal places.

1 $f(x) = x^3 - 7$, $x_0 = 2$

2 $f(x) = x^4 - x - 3$, $x_0 = 1.5$

3 $f(x) = e^x - 4x$, $x_0 = 2$ (Lesson 2)

4 $f(x) = \ln 2x + x - 3$, $x > 0$, $x_0 = 1.5$

5 $f(x) = 2\cos x - x$ (x in radians), $x_0 = 1$

6 (a) Use the Newton–Raphson method on $f(x) = x^3 + 3x^2 - 2$ with $x_0 = 1$ to complete the table (4 d.p.).
(b) Show that $f(x) = (x+1)(x^2 + 2x - 2)$ and find the exact value the iterates approach.

n	0	1	2	3	4
x_n	1				

B · Stating the recurrence Write down the Newton–Raphson recurrence, then use it to find x_1 to 4 decimal places.

7 $f(x) = x^5 - 4x + 1$, $x_0 = 1.3$

8 $f(x) = e^{2x} - 6x$, $x_0 = 1$

9 $f(x) = x^2 - 3\cos x$ (x in radians), $x_0 = 1$

10 $f(x) = \dfrac{3}{x} - \ln x$, $x > 0$, $x_0 = 2.5$

C · Reasoning Explain fully. Q11 and Q13: no calculator.

11 (a) Show that the Newton–Raphson recurrence for $f(x) = x^2 - 10$ can be written as $x_{n+1} = \frac{1}{2}\left(x_n + \frac{10}{x_n}\right)$.
(b) Using $x_0 = 3$, find x_1 and x_2 as fractions.
(c) How close is x_2 to $\sqrt{10}$?

12 $f(x) = \dfrac{x}{\sqrt{1+x^2}}$ has the root 0 (Explore Q4).
(a) Show that $f'(x) = \dfrac{1}{(1+x^2)^{3/2}}$.
(b) Hence show that the Newton–Raphson recurrence is $x_{n+1} = -x_n^{\,3}$.
(c) Describe what happens from $x_0 = 0.5$, from $x_0 = 1$ and from $x_0 = 1.2$.

13 $f(x) = (x-2)^2$ has the root 2.
(a) Explain why a change of sign cannot be used to locate this root.
(b) Show that the Newton–Raphson recurrence is $x_{n+1} = \dfrac{x_n + 2}{2}$.
(c) Using $x_0 = 3$, find x_1, x_2, x_3. How quickly do they approach 2?

14 $f(x) = x^3 - 12x + 5$ has three roots, near -3.66, 0.42 and 3.23.
(a) Find the x-coordinates of the stationary points.
(b) Using $x_0 = 1.9$, find x_1. Explain why it is far from x_0.
(c) From $x_0 = 1.9$ the iterates go to -3.656. Choose a better x_0 for the root near 3.23, and find x_1.

D · Show that

15 $f(x) = x^3 + 2x^2 - 4$.
(a) Show that $f(x) = 0$ has a root α in $[1.1, 1.2]$.
(b) Using $x_0 = 1$, find x_1 and x_2 by Newton–Raphson (4 d.p.).
(c) Show that $\alpha = 1.130$ to 3 d.p.

16 $f(x) = e^x - 2x - 2$.
(a) Show that $f(x) = 0$ has a root α in $[1.6, 1.7]$.
(b) Using $x_0 = 2$, find x_1 and x_2 by Newton–Raphson (4 d.p.).
(c) Show that $\alpha = 1.678$ to 3 d.p.

17 $f(x) = x^2 - 2\sin x - 1$ (x in radians).
(a) Show that $f(x) = 0$ has a root α in $[1.7, 1.8]$.
(b) Using $x_0 = 1.5$, find x_1 and x_2 by Newton–Raphson (4 d.p.).
(c) Show that $\alpha = 1.725$ to 3 d.p.

E · Exam-style question AQA-style (AQA starts from x_1). Use the mark scheme to self-assess.

18 $f(x) = x^3 - 3x^2 + 1$. The equation $f(x) = 0$ has a root α between 2 and 3.
(a) Explain why the Newton–Raphson method fails if $x_1 = 2$ is used as a first approximation to α. [2 marks]
(b) Taking $x_1 = 3$, use the Newton–Raphson method to find x_2 and x_3, giving x_3 to three decimal places. [3 marks]
(c) Show that $\alpha = 2.879$ to three decimal places. [3 marks]