Numerical Methods in Context

Specification: AQA 7357: I4 (with I1, I2) · Edexcel 9MA0: 9.5 (with 9.1–9.3)

INDEPENDENT You do

A · Using a model Answer in context, with units.

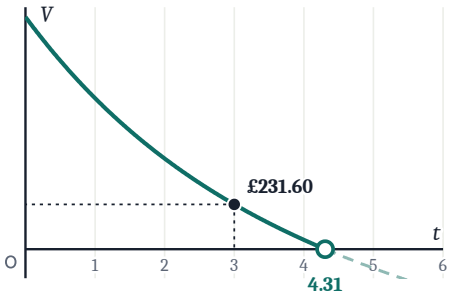
1 The value £V of a laptop t years after it is bought is $V = 1200(0.7)^t - 60t$.
(a) Find its value after 3 years.
(b) Show that the model predicts it is worthless between 4 and 5 years.

(a) $V(3) = 1200(0.7)^3 - 60 \times 3 = 231.6$: £231.60

substitute t = 3

(b) $V(4) = 48.12 > 0$, $V(5) = -98.3 < 0$: change of sign, V continuous ⇒ worthless between 4 and 5 years.

continuous ⇒ root



£231.60 at 3 years; worthless at 4.31

2 The depth of water is $D = 5 + 2\sin(0.5t) + 0.1t$ m, t hours after midnight.
(a) Find the depth at 3 am.
(b) Show that the depth first falls to 4 m between 8 am and 9 am.

(a) $D(3) = 5 + 2\sin(0.5 \times 3) + 0.1 \times 3 = 7.29$ m

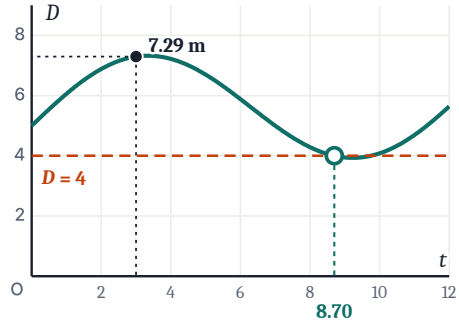
substitute t = 3 (radians)

(b) $f(t) = 1 + 2\sin(0.5t) + 0.1t = D - 4$: $f(8) = 0.286 > 0$, $f(9) = -0.0551 < 0$

f(t) = D − 4

Change of sign, f continuous ⇒ $D = 4$ between 8 am and 9 am.

continuous ⇒ root



D = 4 first at t = 8.70

3 A herd of deer has $P = 300 + 120\sin(0.4t) - 15t$ animals, t months after a count.
(a) Find the number of deer after 2 months.
(b) Show that the herd falls to 200 between $t = 7$ and $t = 8$.

(a) $P(2) = 300 + 120\sin(0.4 \times 2) - 15 \times 2 = 356.08$: 356 deer

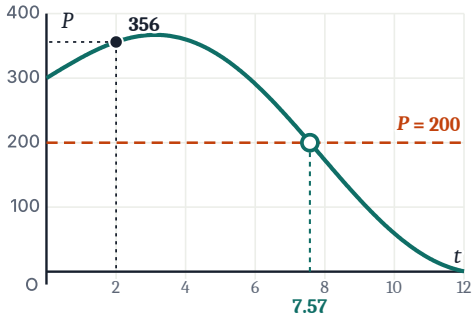
substitute t = 2 (radians)

(b) $f(t) = P - 200$: $f(7) = 35.2 > 0$, $f(8) = -27.0 < 0$

f(t) = P − 200

Change of sign, f continuous ⇒ $P = 200$ between 7 and 8 months.

continuous ⇒ root



356 deer at 2 months; 200 at t = 7.57

4 An engine is switched off. Its temperature is $T = 20 + 70e^{-0.2t} + 0.8t$ °C, t minutes later.
(a) Find its temperature after 5 minutes.
(b) Show that it cools to 40 °C between $t = 8$ and $t = 9$.

(a) $T(5) = 20 + 70e^{-0.2 \times 5} + 0.8 \times 5 = 49.8$ °C

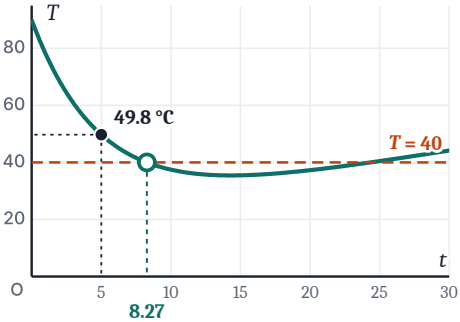
substitute t = 5

(b) $f(t) = T - 40$: $f(8) = 0.533 > 0$, $f(9) = -1.23 < 0$

f(t) = T − 40

Change of sign, f continuous ⇒ $T = 40$ between 8 and 9 minutes.

continuous ⇒ root



49.8 °C at 5 min; 40 °C at 8.27 min

5 A shop spends £x thousand on advertising. Its profit is £P thousand, where $P = 12\ln(x+1) - 1.5x - 2$, $x \geq 0$.

(a) Find P when $x = 0$. What does it mean?

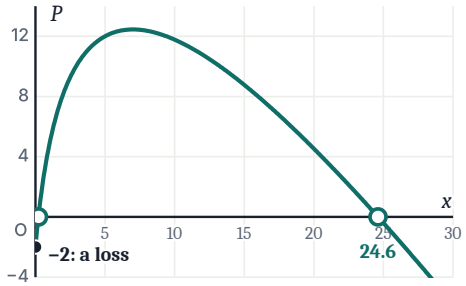
(b) Show that the shop breaks even ($P = 0$) for some x between 24 and 25.

(a) $P(0) = 12\ln(0+1) - 1.5 \times 0 - 2 = -2$: with no advertising the shop makes a loss of £2000.

substitute x = 0

(b) $P(24) = 0.627 > 0$, $P(25) = -0.403 < 0$: change of sign, P continuous ⇒ it breaks even when it spends between £24 000 and £25 000.

continuous ⇒ root



a loss at x = 0; break even at 24.6

6 The medicine in Worked example 5: $C = 6te^{-0.5t}$ mg/L.

(a) Complete the table (2 d.p.).

(b) Between which whole numbers of hours does C rise past 2 mg/L, and fall back below it?

t	0	1	2	3	4	5	6	7
C	0	3.64	4.41	4.02	3.25	2.46	1.79	1.27

(a) Substitute each t (click the cells to check), e.g. $C(1) = 6 \times 1 \times e^{-0.5 \times 1} = 3.64$

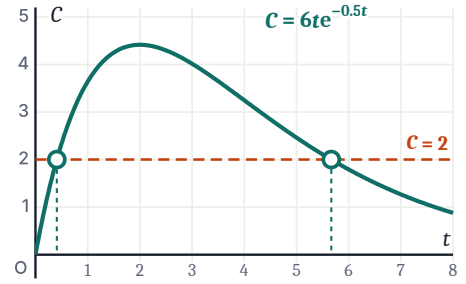
substitute each t

(b) 0 < 2 < 3.64: it rises past 2 between 0 and 1 hours.

compare with 2

2.46 > 2 > 1.79: it falls below 2 between 5 and 6 hours.

compare with 2



past 2 between 0 and 1, and between 5 and 6

B · Setting up Write the equation first, then use a method.

7 A car is worth £V thousand, where $V = 16(0.82)^t - 0.6t$, t years after it is bought.
(a) Write an equation $f(t) = 0$ for the time when it is worth £5000.
(b) Find $f'(t)$.
(c) Using $t_0 = 4$, apply the Newton–Raphson method once (3 d.p.). Give the answer in years and months.

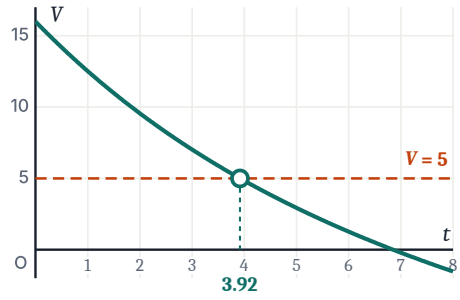
(a) $f(t) = 16(0.82)^t - 0.6t - 5 = 0$ — £5000: V = 5

(b) $f'(t) = 16(0.82)^t \ln 0.82 - 0.6$ — d/dt $a^t = a^t \ln a$

(c) $t_1 = 4 - \dfrac{(-0.1661\ldots)}{-2.036\ldots} = 3.918$ — N–R formula: t_0

0.918 × 12 = 11.0 months: about 3 years 11 months.

years → months: × 12



£5000 at t = 3.92 years

8 The engine in Q4: $T = 20 + 70e^{-0.2t} + 0.8t$.
(a) Find $T'(t)$.
(b) Find the exact time when the temperature is lowest, and the lowest temperature (1 d.p.).
(c) Was a numerical method needed? Explain.

(a) $T'(t) = -14e^{-0.2t} + 0.8$ — differentiate

(b) $14e^{-0.2t} = 0.8 \Rightarrow e^{-0.2t} = \dfrac{2}{35} \Rightarrow t = 5 \ln 17.5 = 14.3$ min

T′(t) = 0: take ln

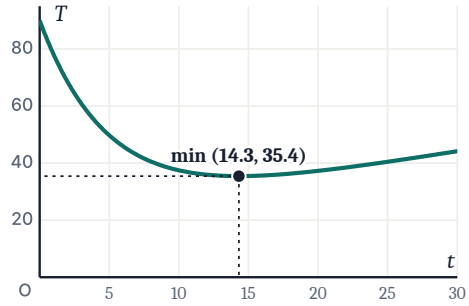
$T''(t) = 2.8e^{-0.2t} > 0$: a minimum.

$T = 20 + 70 \times \dfrac{2}{35} + 0.8 \times 5 \ln 17.5 = 35.4$ °C

T″ > 0; substitute

(c) No: $T'(t) = 0$ can be solved exactly with ln. Use a numerical method only when you cannot.

exact if you can



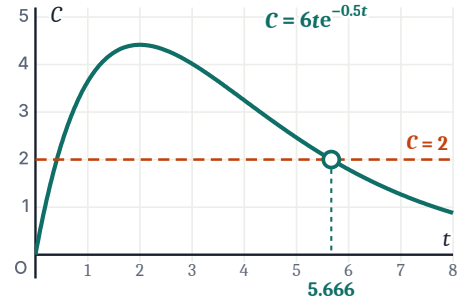
lowest: 35.4 °C at 14.3 min

9 The medicine in Worked example 5: $C = 6te^{-0.5t}$.
(a) Show that $C = 2$ can be written as $t = 2 \ln 3t$.
(b) Using $t_0 = 5$, find t_1, t_2, t_3 (3 d.p.).

(a) $6te^{-0.5t} = 2 \Rightarrow e^{0.5t} = 3t \Rightarrow 0.5t = \ln 3t \Rightarrow t = 2 \ln 3t$ — × e^(0.5t), take ln

(b) $t_1 = 2 \ln(3 \times 5) = 5.416$ — substitute t_0

$t_2 = 5.576$, $t_3 = 5.634$ — ANS key: full value



the iterates approach 5.666

10 The drone in Worked example 2: $h = 30 \sin(0.1t) + 2t - 0.2t^2$ (t in radians). Write down the Newton–Raphson recurrence for $h(t) = 0$. Use it with $t_0 = 18$ to find t_1 (4 d.p.). What does t_1 estimate?

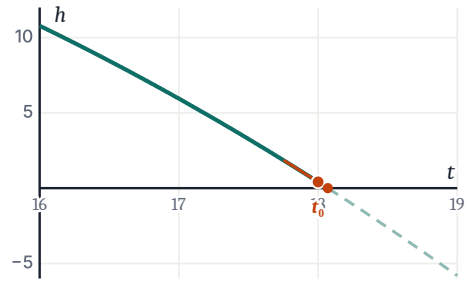
$h'(t) = 3\cos(0.1t) + 2 - 0.4t$ — differentiate

$t_{n+1} = t_n - \dfrac{30 \sin(0.1t_n) + 2t_n - 0.2t_n^2}{3\cos(0.1t_n) + 2 - 0.4t_n}$ — N–R formula

$t_1 = 18 - \dfrac{0.4154\ldots}{-5.882\ldots} = 18.0706$ — substitute t_0 (radians)

An estimate of the time when the drone lands.

in context



one tangent from $t_0 = 18$: the landing time

C · Reasoning Explain in context.

11 The deer in Q3: $P = 300 + 120\sin(0.4t) - 15t$.
(a) Find P when $t = 12.5$ and when $t = 24$.
(b) Criticise the model.
(c) The model first gives $P = 0$ at $t = 12.04$. Suggest a range of validity.

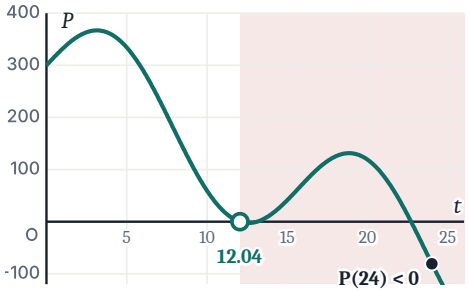
(a) $P(12.5) = -2.57$, $P(24) = -80.92$ — substitute (radians)

(b) A negative number of deer is impossible. The $-15t$ term makes the herd shrink for ever; the sine term only adds a regular rise and fall.

criticise the model

(c) Use it only for $0 \le t \le 12$ months (before it reaches 0).

range of validity



negative from t = 12.04: not valid

12 The shop in Q5: $P = 12\ln(x + 1) - 1.5x - 2$ (£ thousand). $P = 0$ when $x = 0.213$ and when $x = 24.611$.
(a) What does each value mean?
(b) Find the exact spending that gives the greatest profit, and that profit to the nearest £.

(a) Below about £213 of advertising the shop makes a loss; at £213 it breaks even.

in context

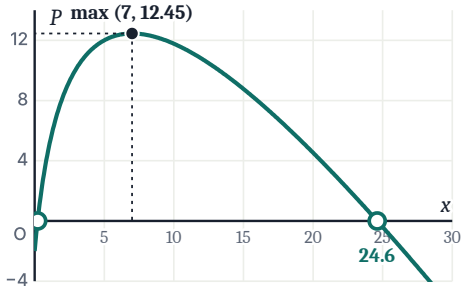
Spending more than about £24 600 costs more than it earns: a loss again.

in context

(b) $P'(x) = \dfrac{12}{x+1} - 1.5 = 0 \Rightarrow x + 1 = 8 \Rightarrow x = 7$ — P'(x) = 0

$P(7) = 12\ln 8 - 12.5 = 12.453$: a profit of £12 453.

substitute x = 7



greatest profit at x = 7

13 The medicine in Worked example 5 ($f(t) = C - 2$). A pupil uses Newton–Raphson with $t_0 = 2$ to find when $C = 2$.
(a) Explain why $t_0 = 2$ fails.
(b) Another pupil uses $t_0 = 2.2$. Find t_1 and explain why it is not useful.
(c) Suggest a good t_0 for each time.

(a) $f'(2) = 6e^{-1}(1 - 1) = 0$: $t = 2$ is the peak. The tangent is flat: no t_1.

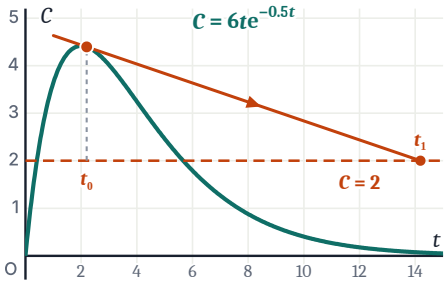
f'(t₀) = 0: fails

(b) $t_1 = 2.2 - \dfrac{2.394\ldots}{-0.1997\ldots} = 14.1861$: the tangent near the peak is nearly flat, so t_1 lands far away, long after the medicine has worn off.

f' small: far away

(c) Read the starts from a sketch: $t_0 = 0.5$ for the first time and $t_0 = 5.5$ for the second.

choose t₀ from the graph



nearly flat tangent at 2.2: t₁ = 14.19

14 The depth model in Q2: $D = 5 + 2\sin(0.5t) + 0.1t$. The depth is 4 m when $t = 8.6996$ and $t = 9.7447$.
(a) Give these as clock times.
(b) For how long is the depth under 4 m?
(c) A sailor says the water is lowest at 9 am. Use $D'(9)$ and $D'(9.5)$ to comment.

(a) 8.6996 → 08:42, 9.7447 → 09:45 — hours → clock times

(b) 9.7447 − 8.6996 = 1.0451 h ≈ 1 h 3 min

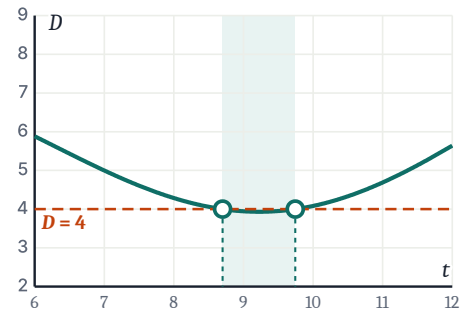
subtract: hours → minutes

(c) $D'(t) = \cos(0.5t) + 0.1$: $D'(9) = -0.111 < 0$, $D'(9.5) = 0.138 > 0$

differentiate, substitute

Still falling at 9 am; the lowest point is between 09:00 and 09:30. The sailor is not quite right.

D' changes sign



under 4 m from 08:42 to 09:45

D · Show that

15 The value of a phone, £V, t years after it is bought, is $V = 900(0.75)^t - 40t$.

(a) Show that it is worthless between 5 and 6 years.

(b) Find $V'(t)$. Using $t_0 = 5$, apply the Newton–Raphson method twice (4 d.p.).

(c) Show that this time is 5.14 years to 2 d.p.

(d) Give it in years and months.

(a) $V(5) = 13.6 > 0$, $V(6) = -79.8 < 0$: change of sign, V continuous ⇒ worthless between 5 and 6 years.

continuous ⇒ root

(b) $V'(t) = 900(0.75)^t \ln 0.75 - 40$ d/dt $a^t = a^t \ln a$

$t_1 = 5 - \dfrac{13.57\ldots}{-101.4\ldots} = 5.1338$ N–R formula: t_0

$t_2 = 5.1338\ldots - \dfrac{0.1562\ldots}{-99.12\ldots} = 5.1354$ substitute t_1: full value

(c) $V(5.135) = 0.0386 > 0$, $V(5.145) = -0.952 < 0$

bounds: ±0.005



(c) worthless at t = 5.14

Change of sign, V continuous ⇒ $5.135 < t < 5.145$, so $t = 5.14$ (2 d.p.)

continuous ⇒ root

(d) $0.1354 \times 12 = 1.6$ months: about 5 years 2 months.

years → months: × 12

16 The cross-section of a channel is $y = (x - 8)\ln(0.5x + 1)$, $0 \le x \le 8$: x m from one bank, y m from the water surface ($y < 0$ is under water).

(a) Find y'. Show that the deepest point is between $x = 3.1$ and $x = 3.2$.

(b) Show that $y' = 0$ can be written as $x = \dfrac{10}{1 + \ln(0.5x + 1)} - 2$.

(c) Using $x_0 = 3$, find x_1, x_2, x_3 (3 d.p.).

(d) The iterates approach 3.14. Find the greatest depth (2 d.p.).

(a) $y' = \ln(0.5x + 1) + \dfrac{x - 8}{x + 2}$ product rule, chain rule

$y'(3.1) = -0.0247 < 0$, $y'(3.2) = 0.0324 > 0$: y' continuous, − to +: a minimum of y, the deepest point.

y' changes sign

(b) $(x + 2)(1 + \ln(0.5x + 1)) = 10 \Rightarrow x = \dfrac{10}{1 + \ln(0.5x + 1)} - 2$

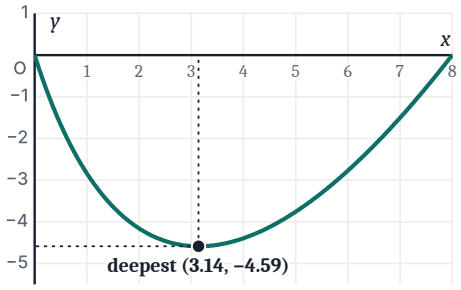
× (x + 2), add x + 2

(c) $x_1 = \dfrac{10}{1 + \ln(0.5 \times 3 + 1)} - 2 = 3.218$ substitute x_0

$x_2 = 3.105$, $x_3 = 3.163$ ANS key: full value

(d) $y(3.14) = -4.5874$: greatest depth 4.59 m

substitute x = 3.14



deepest 4.59 m at x = 3.14

17 The depth of water at a ferry port, t hours after noon, is $D = 6 + 2.4\sin(0.5t) - 0.2t$ m, $0 \le t \le 12$. The ferry needs 5 m.

(a) Show that the depth first falls to 5 m between $t = 6$ and $t = 7$.

(b) Using $t_0 = 6$, apply the Newton–Raphson method once to $f(t) = D - 5$ (4 d.p.).

(c) Show that this time is $t = 6.10$ to 2 d.p.

(d) Give the clock time, to the nearest minute.

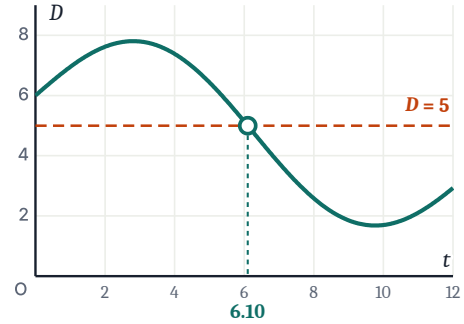
(a) $f(t) = 1 + 2.4\sin(0.5t) - 0.2t$: $f(6) = 0.139 > 0$, $f(7) = -1.24 < 0$: change of sign, f continuous ⇒ a root. — f(t) = D − 5

(b) $f'(t) = 1.2\cos(0.5t) - 0.2$: $t_1 = 6 - \dfrac{0.1387\ldots}{-1.388\ldots} = 6.0999$ — N–R formula: t_o

(c) $f(6.095) = 0.00649 > 0$, $f(6.105) = -0.00746 < 0$ — bounds: ±0.005

Change of sign, f continuous ⇒ $6.095 < t < 6.105$, so $t = 6.10$ (2 d.p.) — continuous ⇒ root

(d) 0.0999 × 60 = 6.00 min: at about 18:06. — hours → clock time



D = 5 first at t = 6.10: 18:06

E · Exam-style question AQA-style. Use the mark scheme to self-assess.

18 A ball is thrown from ground level. Its height h metres, t seconds later, is modelled by

$h(t) = 20\sin(\frac{t}{5}) - 4\cos(\frac{t}{5}) + 4 - 0.5t^2$, $t \ge 0$.

The ball lands when $t = T$.

(a) Show that T is a solution of $t = \sqrt{8 + 40\sin(\frac{t}{5}) - 8\cos(\frac{t}{5})}$. [2 marks]

(b) Using $t_{n+1} = \sqrt{8 + 40\sin(\frac{t_n}{5}) - 8\cos(\frac{t_n}{5})}$ with $t_0 = 6$, find t_1, t_2 and t_3, giving your answers to three decimal places. [3 marks]

(c) Find $h'(t)$. Taking 6 as a first approximation, apply the Newton–Raphson method once to obtain a second approximation for T. Give your answer to three decimal places. [3 marks]

(a) $0.5t^2 = 20\sin(\frac{t}{5}) - 4\cos(\frac{t}{5}) + 4 \Rightarrow$

$t^2 = 8 + 40\sin(\frac{t}{5}) - 8\cos(\frac{t}{5})$ **M1** — h = 0: × 2

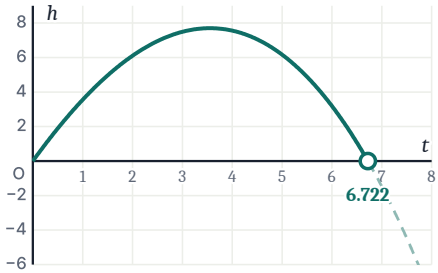
$t > 0$, so $T = \sqrt{8 + 40\sin(\frac{T}{5}) - 8\cos(\frac{T}{5})}$ **A1** — positive root

(b) $t_1 = \sqrt{8 + 40\sin(\frac{6}{5}) - 8\cos(\frac{6}{5})} = 6.510$ **M1 A1** — substitute t_o (radians)

$t_2 = 6.666$, $t_3 = 6.708$ **A1** — ANS key: full value

(c) $h'(t) = 4\cos(\frac{t}{5}) + 0.8\sin(\frac{t}{5}) - t$ **B1** — differentiate: chain rule

$T = 6 - \dfrac{3.191\ldots}{-3.805\ldots} = 6.839$ **M1 A1** — N–R formula: t_o



lands at T = 6.722 s