The Trapezium Rule

Specification: AQA 7357: I3 (with H4, I4) · Edexcel 9MA0: 9.4 (with 9.5)

INDEPENDENT ## You do

A · Using the rule Find h first. Ends once, middle twice.

1 (a) $\int_1^4 f(x)\,dx$ is estimated with 6 strips. Write down h and the x-values.

(b) $\int_0^3 g(x)\,dx$ is estimated using seven ordinates. How many strips are there? Write down h and the x-values.

(a) $h = \frac{4-1}{6} = 0.5$: $x = 1, 1.5, 2, 2.5, 3, 3.5, 4$ h = (b − a) ÷ n

(b) Seven ordinates: 6 strips. $h = \frac{3-0}{6} = 0.5$:
$x = 0, 0.5, 1, 1.5, 2, 2.5, 3$

strips = ordinates − 1

2 Use the trapezium rule with 4 strips to estimate $\int_0^2 \sqrt{4+x^2}\,dx$. Give your answer to 3 d.p.

$h = (2-0) \div 4 = 0.5$: $x = 0, 0.5, 1, 1.5, 2$ h = (b − a) ÷ n

$y_0 = 2$, $y_1 = 2.0616$, $y_2 = 2.2361$, $y_3 = 2.5$, $y_4 = 2.8284$

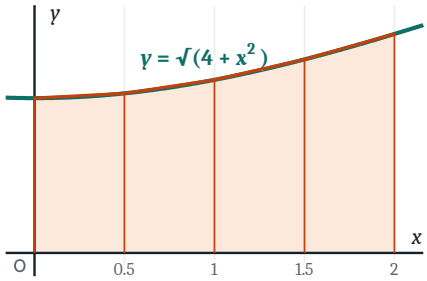
ordinates: 4 d.p.

$\int_0^2 \sqrt{4+x^2}\,dx \approx ½ \times 0.5 \times [2 + 2(2.0616 + 2.2361 + 2.5) + 2.8284]$

ends once, middle twice

$= 0.25 \times 18.4238 = 4.60595 \approx 4.606$ (3 d.p.)

round at the end



4 trapezia: about 4.606

3 Use the trapezium rule with five ordinates to estimate $\int_0^1 \frac{1}{1+x^3}\,dx$. Give your answer to 4 s.f.

$h = (1-0) \div 4 = 0.25$: $x = 0, 0.25, 0.5, 0.75, 1$

5 ordinates: 4 strips

$y_0 = 1$, $y_1 = 0.9846$, $y_2 = 0.8889$, $y_3 = 0.7033$, $y_4 = 0.5$

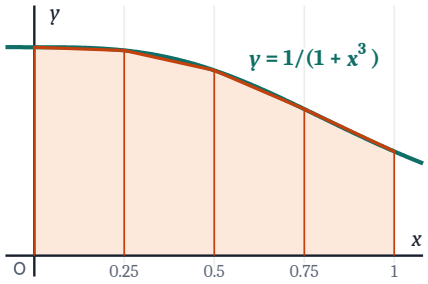
ordinates: 4 d.p.

$\int_0^1 \frac{1}{1+x^3}\,dx \approx ½ \times 0.25 \times [1 + 2(0.9846 + 0.8889 + 0.7033) + 0.5]$

ends once, middle twice

$= 0.125 \times 6.6536 = 0.8317 \approx 0.8317$ (4 s.f.)

round at the end



five ordinates: 4 strips

4 Use the trapezium rule with 4 strips to estimate $\int_0^1 \sin(x^2)\,dx$ (x in radians). Give your answer to 3 s.f.

$h = (1-0) \div 4 = 0.25$: $x = 0, 0.25, 0.5, 0.75, 1$ h = (b − a) ÷ n

$y_0 = 0$, $y_1 = 0.0625$, $y_2 = 0.2474$, $y_3 = 0.5333$, $y_4 = 0.8415$

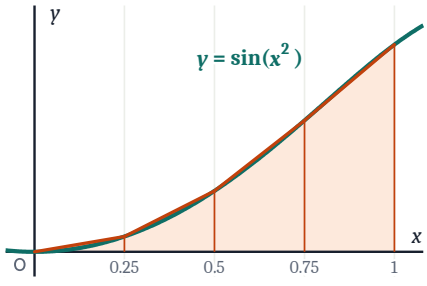
ordinates (radians)

$\int_0^1 \sin(x^2)\,dx \approx ½ \times 0.25 \times [0 + 2(0.0625 + 0.2474 + 0.5333) + 0.8415]$

ends once, middle twice

$= 0.125 \times 2.5279 = 0.3159875 \approx 0.316$ (3 s.f.)

round at the end



4 strips (radians)

5 The table shows some values of $y = \ln(1 + \sqrt{x})$.

(a) Complete the table (4 d.p.).

(b) Use the trapezium rule with all the values to estimate $\int_0^3 \ln(1 + \sqrt{x})\,dx$ (3 s.f.).

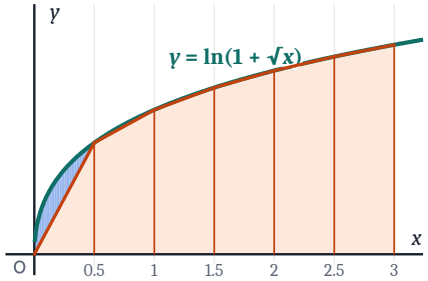
x	0	0.5	1	1.5	2	2.5	3
y	0	0.5348	0.6931	0.7996	0.8814	0.9482	1.0051

(a) $y = \ln(1 + \sqrt{1.5}) = 0.7996$, $y = \ln(1 + \sqrt{2.5}) = 0.9482$ — substitute each x

(b) $h = 0.5$ (7 ordinates: 6 strips) — h: gap between ordinates

$\frac{1}{2} \times 0.5 \times [0 + 2(0.5348 + 0.6931 + 0.7996 + 0.8814 + 0.9482) + 1.0051]$ — ends once, middle twice

$= 0.25 \times 8.7193 = 2.179825 \approx 2.18$ (3 s.f.) — round at the end



6 strips of width 0.5

6 (a) Complete the table for $y = \frac{1}{\sqrt{1 + x^2}}$ (4 d.p.).

(b) Use the trapezium rule with all the values to estimate $\int_0^2 \frac{1}{\sqrt{1 + x^2}}\,dx$ (3 d.p.).

x	0	0.5	1	1.5	2
y	1	0.8944	0.7071	0.5547	0.4472

(a) $y = \frac{1}{\sqrt{1 + 0.5^2}} = 0.8944$; the others the same way (click the cells).

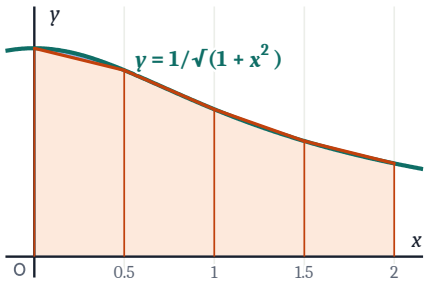
substitute each x

(b) $h = 0.5$ — h: gap between ordinates

$\frac{1}{2} \times 0.5 \times [1 + 2(0.8944 + 0.7071 + 0.5547) + 0.4472]$

ends once, middle twice

$= 0.25 \times 5.7596 = 1.4399 \approx 1.440$ (3 d.p.) — round at the end



4 strips of width 0.5

B · Over or under? Better estimates Decide from the shape: f''(x).

7 (a) Find the exact value of $\int_1^4 \frac{1}{\sqrt{x}}\,dx$.

(b) Use the trapezium rule with 3 strips to estimate it (3 s.f.).

(c) Find the percentage error (2 s.f.).

(d) Explain why it is an overestimate.

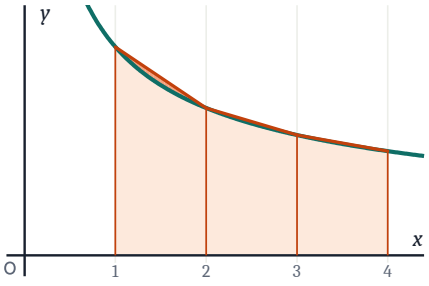
(a) $[2x^{½}]$ from 1 to 4 $= 4 - 2 = 2$ — integrate: add 1 to the power

(b) $h = 1$, $y_0 = 1$, $y_1 = 0.7071$, $y_2 = 0.5774$, $y_3 = 0.5$ — h, then the ordinates

$½ \times 1 \times [1 + 2(0.7071 + 0.5774) + 0.5]$
$= 0.5 \times 4.069 = 2.0345 \approx 2.03$ (3 s.f.) — trapezium rule (3 s.f.)

(c) $|2.0345 - 2| \div 2 \times 100 = 1.7\%$ — |T − I| ÷ I × 100

(d) $f''(x) = ¾x^{-5/2} > 0$ for $1 \le x \le 4$: **convex**, chords above the curve: an **over**estimate. — convex ⇒ overestimate



convex: chords above (over)

8 (a) Find the exact value of $\int_0^2 x^3\,dx$.

(b) Use the trapezium rule with 2 strips, and then with 4 strips.

(c) Find the error in each. What happens to the error when the number of strips doubles?

(d) Explain why both are overestimates.

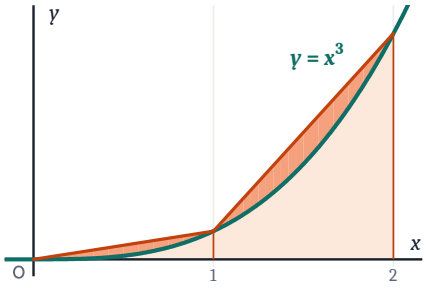
(a) $[\frac{x^4}{4}]$ from 0 to 2 $= 4$ — integrate

(b) $n = 2$: $½ \times 1 \times [0 + 2(1) + 8] = 0.5 \times 10 = 5$ — trapezium rule: n = 2

$n = 4$: $½ \times 0.5 \times [0 + 2(0.125 + 1 + 3.375) + 8]$
$= 0.25 \times 17 = 4.25$ — trapezium rule: n = 4

(c) Errors 1 and 0.25: doubling the strips divides the error by 4. — compare the errors

(d) $f''(x) = 6x \ge 0$ for $0 \le x \le 2$: **convex**, chords above the curve: an **over**estimate. — convex ⇒ overestimate



2 strips: chords well above

9 (a) Explain why $\int_0^3 \sqrt{9 - x^2}\,dx = \frac{9\pi}{4}$.

(b) Use the trapezium rule with 3 strips to estimate it (3 s.f.).

(c) Find the percentage error.

(d) Is it an over- or underestimate? Why is the error so large?

(a) $y = \sqrt{9 - x^2}$ is the top half of $x^2 + y^2 = 9$. From 0 to 3 it is a quarter circle of radius 3: $\frac{1}{4} \times \pi \times 3^2 = 9\pi/4$

area of a quarter circle

(b) $h = 1, y_0 = 3, y_1 = 2.8284, y_2 = 2.2361, y_3 = 0$

h, then the ordinates

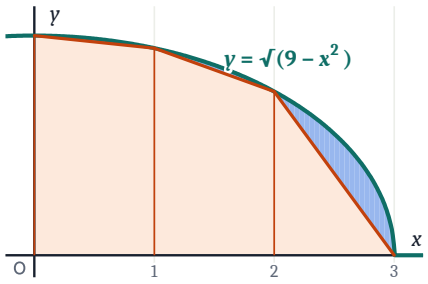
$\frac{1}{2} \times 1 \times [3 + 2(2.8284 + 2.2361) + 0]$
$= 0.5 \times 13.129 = 6.5645 \approx 6.56$ (3 s.f.)

trapezium rule (3 s.f.)

(c) $|6.5645 - 9\pi/4| \div (9\pi/4) \times 100 = 7.13\%$ $|T - I| \div I \times 100$

(d) Under: concave, chords below. Large because the curve falls steeply to 0 at $x = 3$: the last chord misses a lot.

concave; steep at the end



quarter circle: the last strip misses most

10 $f(x) = x^3 - 3x^2 + 4$.

(a) Find $f''(x)$. Explain why you cannot tell from the shape whether the trapezium rule over- or underestimates $\int_0^3 (x^3 - 3x^2 + 4)\,dx$.

(b) Use the trapezium rule with 3 strips.

(c) Find the exact value. Was your estimate an over- or underestimate?

(a) $f''(x) = 6x - 6$: concave for $x < 1$, convex for $x > 1$. The concavity changes, so the shape cannot tell you.

concavity changes

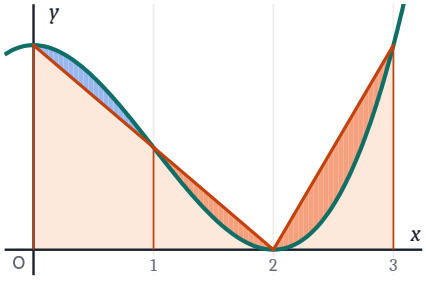
(b) $h = 1, y_0 = 4, y_1 = 2, y_2 = 0, y_3 = 4$ h, then the ordinates

$\frac{1}{2} \times 1 \times [4 + 2(2 + 0) + 4] = 0.5 \times 12 = 6$ trapezium rule

(c) $[\frac{x^4}{4} - x^3 + 4x]$ from 0 to 3 $= \frac{81}{4} - 27 + 12 = 5.25$

integrate

$6 > 5.25$: an **over**estimate, by 0.75. compare



concave then convex: here an overestimate

C · Reasoning Explain each conclusion.

11 $y = 2^x$, $0 \le x \le 2$. Let $I = \int_0^2 2^x \, dx$.

(a) Using 4 strips, find the lower and upper rectangle sums.

(b) Use the trapezium rule with 4 strips.

(c) The curve is convex. Give the closest limits that I must lie between (3 d.p.).

(a) $h = 0.5$, $y_0 = 1$, $y_1 = 1.4142$, $y_2 = 2$, $y_3 = 2.8284$, $y_4 = 4$

h, then the ordinates

Increasing: lower $0.5 \times (1 + 1.4142 + 2 + 2.8284) = 3.6213$; upper $0.5 \times (1.4142 + 2 + 2.8284 + 4) = 5.1213$

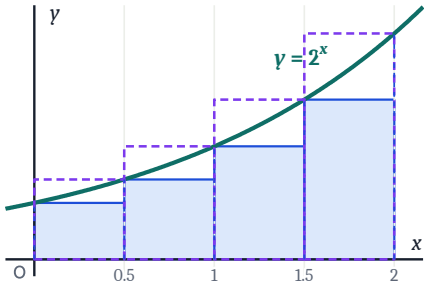
increasing: left ends lower

(b) $\frac{1}{2} \times 0.5 \times [1 + 2(1.4142 + 2 + 2.8284) + 4]$
$= 0.25 \times 17.4852 = 4.3713$

trapezium rule

(c) Convex: T is an overestimate, closer than the upper rectangles. $3.621 < I < 4.372$ (rounded outwards)

lower rectangles, T



lower and upper rectangles

12 (a) Show that the trapezium rule with 4 strips gives $\int_0^2 \sqrt{1 + x^4} \, dx \approx 3.734$.

(b) Hence estimate $\int_0^2 (5\sqrt{1 + x^4} - 3x^2) \, dx$.

(c) Hence estimate $\int_0^2 \sqrt{9 + 9x^4} \, dx$.

(a) $h = 0.5$, $y_0 = 1$, $y_1 = 1.0308$, $y_2 = 1.4142$, $y_3 = 2.4622$, $y_4 = 4.1231$

h, then the ordinates

$\frac{1}{2} \times 0.5 \times [1 + 2(1.0308 + 1.4142 + 2.4622) + 4.1231]$
$= 0.25 \times 14.9375 = 3.734375 \approx 3.734$ (3 d.p.)

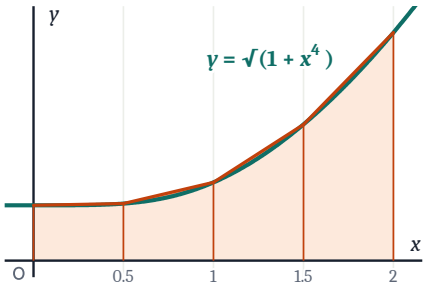
trapezium rule

(b) $5 \times 3.734 - [x^3]$ from 0 to $2 = 18.67 - 8 = 10.670$

split: $5T - \int 3x^2$

(c) $\sqrt{9 + 9x^4} = 3\sqrt{1 + x^4}$: $3 \times 3.734 = 11.202$

take out √9



4 trapezia: about 3.734

13 (a) Use the trapezium rule with 2 strips, then 4 strips, to estimate $\int_0^4 x^2\,dx$.
(b) Find the exact value, and the error in each estimate.
(c) Predict the estimate with 8 strips. Check your prediction.

(a) $n = 2$: $\frac{1}{2} \times 2 \times [0 + 2(4) + 16] = 1 \times 24 = 24$; $n = 4$: $\frac{1}{2} \times 1 \times [0 + 2(1 + 4 + 9) + 16] = 0.5 \times 44 = 22$

trapezium rule: n = 2, 4

(b) $[x^3/3]$ from 0 to 4 $= 64/3$. Errors $24 - 64/3 = 8/3$, $22 - 64/3 = 2/3$

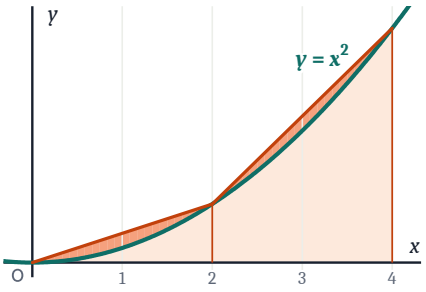
exact, then the errors

(c) Error ÷ 4 each time: $2/3 \div 4 = 1/6$, so predict $64/3 + 1/6 = 21.5$

error ÷ 4

Check: $\frac{1}{2} \times 0.5 \times [0 + 2(0.25 + 1 + 2.25 + 4 + 6.25 + 9 + 12.25) + 16] = 0.25 \times 86 = 21.5$ ✓

trapezium rule: n = 8



2 strips: error 8/3

14 (a) Use the trapezium rule with 4 strips to estimate $\int_0^4 (x^2 - 4x)\,dx$.
(b) Find the exact value. Hence find the area between the curve and the x-axis.
(c) Is (a) an over- or underestimate of the integral? Of the area?

(a) $h = 1$, $y_0 = 0$, $y_1 = -3$, $y_2 = -4$, $y_3 = -3$, $y_4 = 0$; $\frac{1}{2} \times 1 \times [0 + 2((-3) + (-4) + (-3)) + 0] = 0.5 \times (-20) = -10$

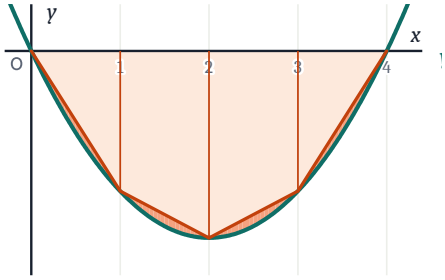
trapezium rule

(b) $[\frac{x^3}{3} - 2x^2]$ from 0 to 4 $= \frac{64}{3} - 32 = -\frac{32}{3}$. The region is below the axis: area $\frac{32}{3} \approx 10.67$

integrate: below the axis

(c) Convex ($f''(x) = 2 > 0$): the chords are above the curve, so $-10 > -\frac{32}{3}$: an **over**estimate of the integral, but $10 < 10.67$: an **under**estimate of the area.

sign matters



below the axis: T = −10

D · In context The area under a rate graph is a total: distance, volume.

15 A car accelerates from rest. Its speed v m/s after t seconds is shown.

t	0	5	10	15	20	25	30
v	0	8.2	13.6	17.1	19.2	20.3	20.8

(a) Estimate the distance it travels in the 30 seconds.
(b) The speed–time graph is concave. Is (a) too big or too small?

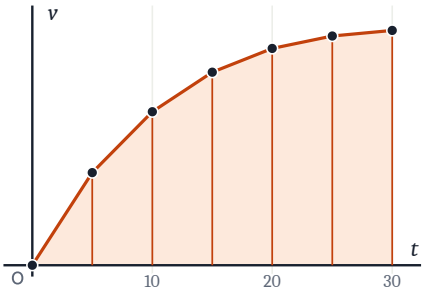
(a) $h = 5$: $\frac{1}{2} \times 5 \times [0 + 2(8.2 + 13.6 + 17.1 + 19.2 + 20.3) + 20.8]$ $= 2.5 \times 177.6 = 444$ m

distance = area under v–t

calculate

(b) Concave: chords below the curve: an **under**estimate (too small).

concave ⇒ underestimate



area under the v–t graph

16 A train's speed is $v = 20(1 - e^{-0.3t})$ m/s, $0 \le t \le 10$.
(a) Use the trapezium rule with 5 strips to estimate the distance it travels in the first 10 s (3 s.f.).
(b) Show that the graph of v is concave. What does this tell you about (a)?

(a) $h = (10 - 0) \div 5 = 2$: $t = 0, 2, 4, 6, 8, 10$ — h = (b − a) ÷ n

$y_0 = 0$, $y_1 = 9.0238$, $y_2 = 13.9761$, $y_3 = 16.6940$, $y_4 = 18.1856$, $y_5 = 19.0043$

ordinates: 4 d.p.

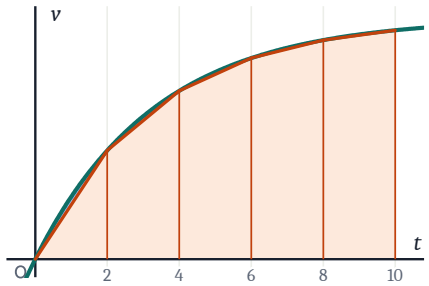
$\int_0^{10} 20(1 - e^{-0.3t})\,dt \approx \frac{1}{2} \times 2 \times [0 + 2(9.0238 + 13.9761 + 16.6940 + 18.1856) + 19.0043]$

ends once, middle twice

$= 1 \times 134.7633 = 134.7633 \approx 135$ (3 s.f.) — distance (3 s.f.)

(b) $v' = 6e^{-0.3t}$, $v'' = -1.8e^{-0.3t} < 0$: concave, so (a) is an **under**estimate.

concave ⇒ underestimate



concave: chords below (under)

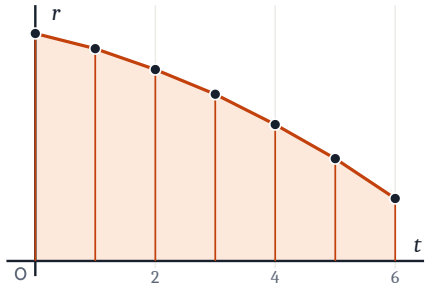
17 Water flows into a tank. The rate r litres per minute, t minutes after a tap is opened, is shown.

t	0	1	2	3	4	5	6
r	12.0	11.2	10.1	8.8	7.2	5.4	3.3

(a) Estimate the volume of water that flows in during the 6 minutes (3 s.f.).
(b) The rate falls faster and faster (the graph is concave). Is (a) too big or too small?

(a) $h = 1$: $\frac{1}{2} \times 1 \times [12.0 + 2(11.2 + 10.1 + 8.8 + 7.2 + 5.4) + 3.3]$ — volume = area under r–t
$= 0.5 \times 100.7 = 50.35 \approx 50.4$ litres — calculate (3 s.f.)

(b) Concave: chords below the curve: 50.4 litres is **too small**. — concave ⇒ underestimate



volume = area under the rate graph

E · Exam-style question AQA-style. Use the mark scheme to self-assess.

18 (a) Use the trapezium rule with five ordinates to find an approximate value for $\int_0^2 \frac{12}{x^2+4}\,dx$, giving your answer to four significant figures. [3 marks]

(b) The curve $y = \frac{12}{x^2+4}$ has a point of inflection at $x = 2/\sqrt{3}$. Explain why you cannot tell from the shape of the curve whether your answer to (a) is an overestimate or an underestimate. [1 mark]

(c) The curve is decreasing for $x > 0$. By considering rectangles, find limits, to 3 s.f., that the value of the integral must lie between. [3 marks]

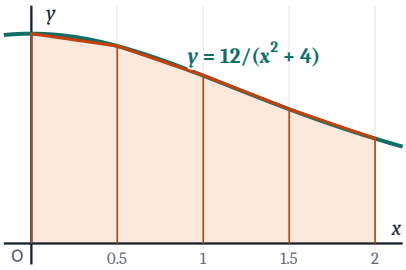
(d) State how the approximation in (a) could be improved. [1 mark]

(a) $h = (2 - 0) \div 4 = 0.5$: $y_0 = 3$, $y_1 = 2.8235$, $y_2 = 2.4$, $y_3 = 1.92$, $y_4 = 1.5$ **B1**

5 ordinates: 4 strips

$\frac{1}{2} \times 0.5 \times [3 + 2(2.8235 + 2.4 + 1.92) + 1.5]$ **M1** — ends once, middle twice

$= 0.25 \times 18.787 = 4.69675 \approx 4.697$ (4 s.f.) **A1** — round at the end



concave, then convex after x = 1.15

(b) Concave, then convex: some chords below the curve, some above, so the shape cannot decide it. **E1**

concavity changes

(c) Lower (right ends) $0.5 \times (2.8235 + 2.4 + 1.92 + 1.5) = 4.32175$; upper (left ends) $0.5 \times (3 + 2.8235 + 2.4 + 1.92) = 5.07175$ **M1 M1**

decreasing: right ends lower limits (3 s.f.)

$4.32 < I < 5.08$ (rounded outwards) **A1**

(d) Use more ordinates (more strips, smaller h). **E1** — more strips