Unit Review

Specification: AQA 7357: I1, I2, I3, I4 · Edexcel 9MA0: 9.1, 9.2, 9.3, 9.4, 9.5

INDEPENDENT Mixed practice

A · Fluency One method each. Which one?

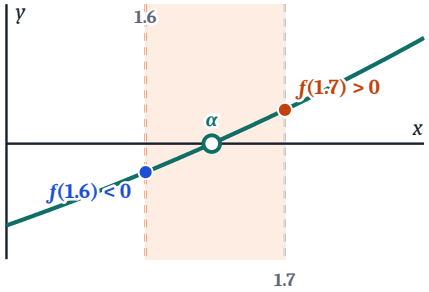
1 $f(x) = 2x^3 - 3x - 4$. Show that $f(x) = 0$ has a root in the interval $[1.6, 1.7]$.

$f(1.6) = -0.608 < 0, f(1.7) = 0.726 > 0$ from L1

substitute both ends

Change of sign, f continuous ⇒ a root in $[1.6, 1.7]$.

continuous ⇒ root



change of sign on [1.6, 1.7]

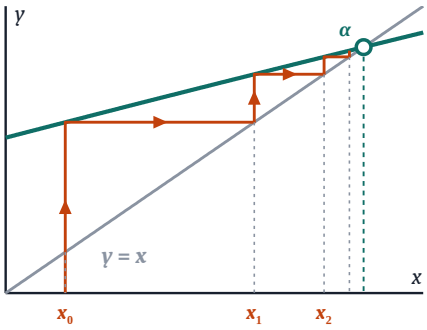
2 Use $x_{n+1} = \ln(3x_n + 2)$ with $x_0 = 2$ to find x_1, x_2, x_3 and x_4, giving each to 4 d.p.

$x_1 = \ln(3 \times 2 + 2) = 2.0794$ from L2 — substitute x_0

$x_2 = 2.1088$, $x_3 = 2.1194$, $x_4 = 2.1233$ — ANS key: full value

Increasing each time, closing in on the root $\alpha = 2.125$ from below.

describe the iterates



staircase up to α

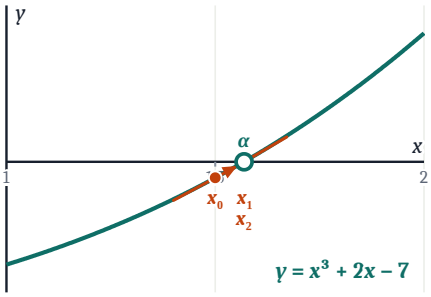
3 $f(x) = x^3 + 2x - 7$. Use the Newton–Raphson method with $x_0 = 1.5$ to find x_1 and x_2, giving each to 4 d.p.

$f'(x) = 3x^2 + 2$ from L4 — differentiate

$f(1.5) = -0.625, f'(1.5) = 8.75$ — substitute x_0 into f and f'

$x_1 = 1.5 - \dfrac{(-0.625)}{8.75} = 1.5714$ — N–R formula

$x_2 = 1.5714\ldots - \dfrac{0.02332\ldots}{9.408\ldots} = 1.5689$ — substitute x_1: full value



two tangents: x_1, x_2

4 The table shows values of $y = f(x)$. Use the trapezium rule with all the values to estimate $\int_0^2 f(x)\,dx$. Give your answer to 3 d.p.

x	0	0.5	1	1.5	2
y	2	2.0310	2.2361	2.7157	3.4641

Five ordinates: 4 strips, $h = 0.5$ from L6

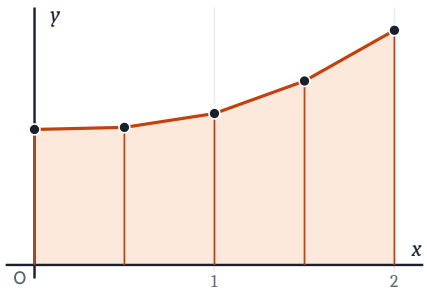
h: gap between ordinates

$\int_0^2 f(x)\,dx \approx \frac{1}{2} \times 0.5 \times [2 + 2(2.0310 + 2.2361 + 2.7157) + 3.4641]$

ends once, middle twice

$= 0.25 \times 19.4297 = 4.857425 \approx 4.857$

round at the end



4 trapezia through the data

5 Without a calculator, use $x_{n+1} = \dfrac{6}{x_n + 1}$ with $x_0 = 1$ to find x_1 to x_4 as exact values.

(a) Is the diagram of the iterates a staircase or a cobweb?

(b) Find the limit α exactly.

$x_1 = \dfrac{6}{2} = 3$, $x_2 = \dfrac{6}{4} = \dfrac{3}{2}$, $x_3 = \dfrac{6}{5/2} = \dfrac{12}{5}$, $x_4 = \dfrac{6}{17/5} = \dfrac{30}{17}$

from L2, L3

substitute, exactly

(a) Up, down, up, down: the values jump from one side of the limit to the other: a **cobweb**, closing in.

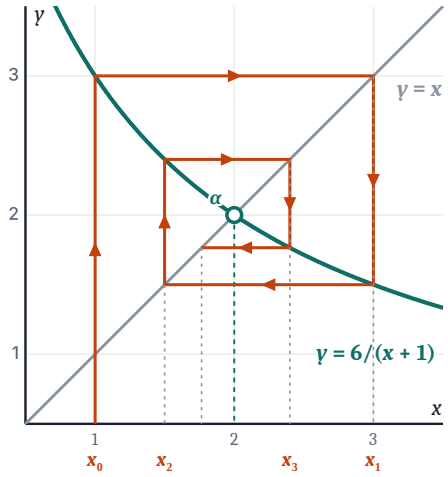
alternate sides: cobweb

(b) $x = \dfrac{6}{x+1} \Rightarrow x^2 + x - 6 = 0 \Rightarrow (x+3)(x-2) = 0$

limit: x = g(x)

All the iterates are positive, so $\alpha = 2$.

reject the negative root



cobweb closing in on 2

6 The depth of water in a harbour is modelled by $d = 3 + 2\sin(0.5t) - 0.1t^2$ metres, t hours after midnight (angles in radians).
Show that the depth is 4 m at some time between 01:00 and 02:00.

Set $F(t) = d - 4 = 2\sin(0.5t) - 0.1t^2 - 1$ from L5, L1

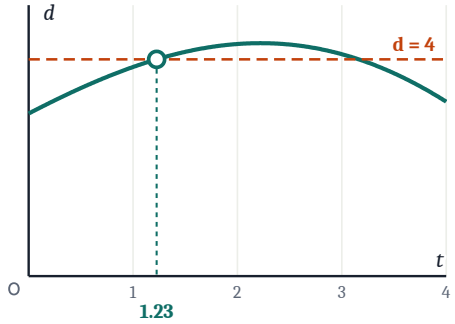
model − 4 = 0

$F(1) = -0.141 < 0$, $F(2) = 0.283 > 0$

substitute t = 1 and t = 2 (radians)

Change of sign and F is continuous, so $d = 4$ at some time between $t = 1$ and $t = 2$: between 01:00 and 02:00.

in context



d = 4 at t = 1.23

B · Multi-step Several methods in one question.

7 (a) Show that $x^3 + 5x - 10 = 0$ has a root α in the interval $[1.4, 1.5]$.

(b) Show that the equation can be written as $x = \dfrac{10}{x^2 + 5}$.

(c) Use $x_{n+1} = \dfrac{10}{x_n^{\,2} + 5}$ with $x_0 = 1.4$ to find x_1, x_2 and x_3 to 4 d.p.

(d) Show that $\alpha = 1.42$ correct to 2 d.p.

(a) $f(1.4) = -0.256 < 0, f(1.5) = 0.875 > 0$ from L1, L2

substitute both ends

Change of sign, f continuous ⇒ a root in $[1.4, 1.5]$.

continuous ⇒ root

(b) $x^3 + 5x = 10 \Rightarrow x(x^2 + 5) = 10 \Rightarrow x = \dfrac{10}{x^2 + 5}$

factorise, ÷ $(x^2 + 5)$

(c) $x_1 = 10/((1.4)^2 + 5) = 1.4368$ substitute x_0

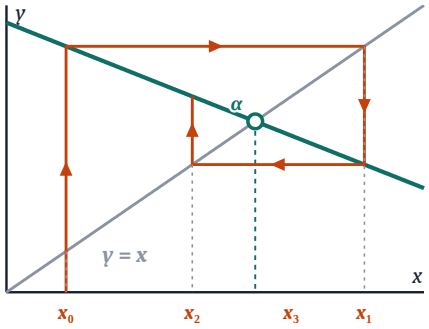
$x_2 = 1.4156$, $x_3 = 1.4278$ ANS key: full value

(d) $f(1.415) = -0.0919 < 0, f(1.425) = 0.0186 > 0$

bounds: ±0.005

Change of sign, f continuous ⇒ $1.415 < \alpha < 1.425$: $\alpha = 1.42$.

continuous ⇒ root



x_1, x_2, x_3 close in on α

8 $f(x) = x^2 - 3\cos x$ (x in radians).

(a) Find $f'(x)$.

(b) Use the Newton–Raphson method with $x_0 = 1.2$ to find x_1 and x_2 to 4 d.p.

(c) Show that the root of $f(x) = 0$ near 1.2 is 1.131 correct to 3 d.p.

(a) $f'(x) = 2x + 3\sin x$ from L4, L1 differentiate: cos → −sin

(b) $f(1.2) = 0.3529\ldots, f'(1.2) = 5.196\ldots$ substitute x_0 into f and f'

$x_1 = 1.2 - \dfrac{0.3529\ldots}{5.196\ldots} = 1.1321$ N–R formula (radians)

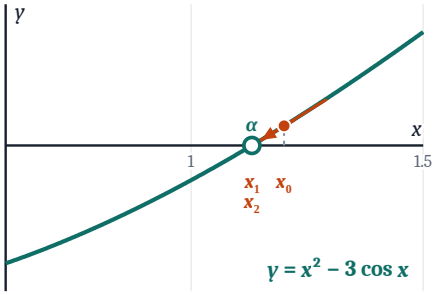
$x_2 = 1.1321\ldots - \dfrac{0.007265\ldots}{4.980\ldots} = 1.1306$ substitute x_1: full value

(c) $f(1.1305) = -0.000592 < 0, f(1.1315) = 0.00438 > 0$

bounds: ±0.0005

Change of sign, f continuous ⇒ $1.1305 < \alpha < 1.1315$: $\alpha = 1.131$.

continuous ⇒ root



two tangents: x_1, x_2

9 (a) Find the exact value of $\int_1^3 (x^2 + 2x^{-2})\,dx$.
(b) Use the trapezium rule with 4 strips to estimate it, to 4 s.f.
(c) Find the percentage error of your estimate.
(d) Explain why your estimate is too big.

(a) $[\frac{x^3}{3} - 2x^{-1}]$ from 1 to 3 $= (9 - \frac{2}{3}) - (\frac{1}{3} - 2) = 10$

from L6

integrate: $x^n \to x^{n+1} \div (n+1)$

(b) $h = (3-1) \div 4 = 0.5$: $x = 1, 1.5, 2, 2.5, 3$ h = (b − a) ÷ n

$y_0 = 3$, $y_1 = 3.1389$, $y_2 = 4.5$, $y_3 = 6.57$, $y_4 = 9.2222$ ordinates: 4 d.p.

$\int_1^3 (x^2 + 2x^{-2})\,dx \approx ½ \times 0.5 \times [3 + 2(3.1389 + 4.5 + 6.57) + 9.2222]$

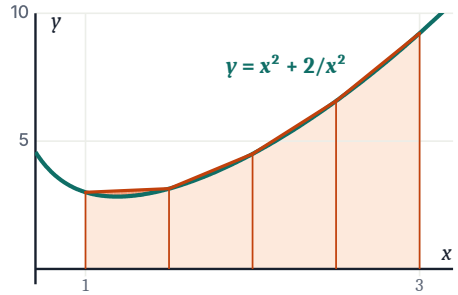
ends once, middle twice

$= 0.25 \times 40.64 = 10.16$ (4 s.f.) round at the end

(c) $|10.16 - 10| \div 10 \times 100 = 1.6\%$ |T − I| ÷ I × 100

(d) $f''(x) = 2 + 12x^{-4} > 0$: convex, so every chord lies above the curve: an **over**estimate.

convex ⇒ overestimate



convex: an overestimate

10 The concentration of a drug in the blood is modelled by $C = 8te^{-0.5t}$ mg/L, t hours after it is taken.
(a) Find the exact greatest concentration.
(b) The drug works while $C \geq 3$. Show that C falls to 3 between $t = 5$ and $t = 6$.
(c) Use the Newton–Raphson method with $t_0 = 5$ to find, to the nearest minute, when C falls to 3.

(a) $C' = 8e^{-0.5t} - 4te^{-0.5t} = 4e^{-0.5t}(2 - t)$ from L5, L1, L4

product rule

$C' = 0 \Rightarrow t = 2$: $C = 16e^{-1} = \frac{16}{e} = 5.89$ mg/L greatest: C′ = 0

(b) Let $F(t) = 8te^{-0.5t} - 3$: $F(5) = 0.283 > 0$, $F(6) = -0.610 < 0$

model − 3 = 0

Change of sign, F continuous: $C = 3$ between $t = 5$ and $t = 6$.

continuous ⇒ root

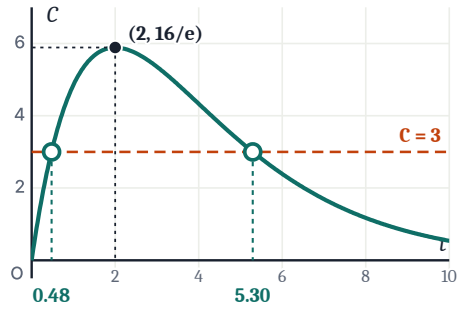
(c) $F'(t) = C' = 4e^{-0.5t}(2 - t)$: $t_1 = 5 - \frac{0.2833...}{-0.9850...} = 5.2877$

N–R formula

$t_2 = 5.2953$, $t_3 = 5.2953$ ANS key: full value

$t \approx 5.2953$ h: $0.2953 \times 60 = 17.7$, so 5 h 18 min after it is taken.

in context: × 60



greatest at t = 2; C = 3 again at 5.30

11 $x^2 + 2x - 4 = 0$ has a positive root $\alpha = \sqrt{5} - 1$.

(a) Use $x_{n+1} = \dfrac{4}{x_n + 2}$ with $x_0 = 1$ to find x_1 to x_4 as fractions. Staircase or cobweb?

(b) Use $x_{n+1} = \dfrac{4 - x_n^{\,2}}{2}$ with $x_0 = 1.2$ to find x_1 to x_4 (4 d.p.). Describe what happens.

(c) Explain, using the diagrams, why one iteration converges to α and the other does not.

(a) $x_1 = \dfrac{4}{3}$, $x_2 = \dfrac{6}{5}$, $x_3 = \dfrac{5}{4}$, $x_4 = \dfrac{16}{13}$ from L2, L3

substitute, exactly

Alternate sides of $\alpha = 1.2361$, closing in: a **cobweb**.

alternate sides: cobweb

(b) $x_1 = (4 - (1.2)^2)/2 = 1.2800$ substitute x_0

$x_2 = 1.1808$, $x_3 = 1.3029$, $x_4 = 1.1513$ ANS key: full value

Alternate sides, further from α each time: a cobweb that **diverges**.

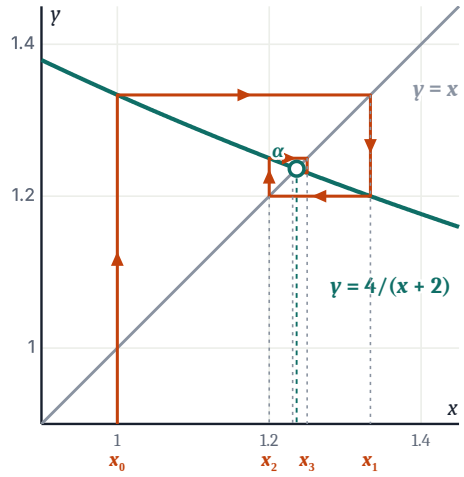
moving away: diverges

(c) Near α, $y = 4/(x + 2)$ is less steep than $y = x$ (gradient -0.38): it closes in.

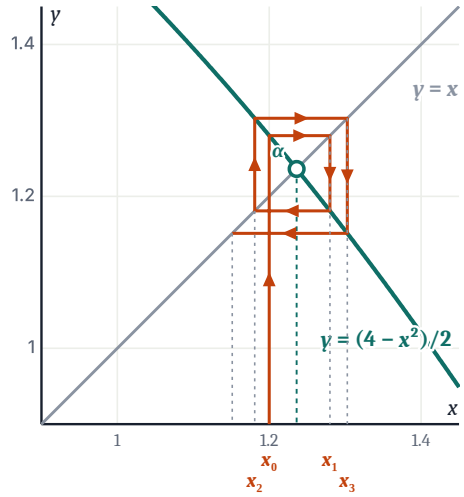
compare with y = x

$y = (4 - x^2)/2$ is steeper (gradient $-\alpha = -1.24$): it spirals out.

steeper than y = x



(a): cobweb closing in



(b): cobweb spiralling out

C · Reasoning Explain. When does a method fail?

12 $f(x) = 4x^2 - 12x + 9$. Sam says: "$f(1) = 1$ and $f(2) = 1$ have the same sign, so $f(x) = 0$ has no root between 1 and 2."
Is Sam right? Explain, without a calculator.

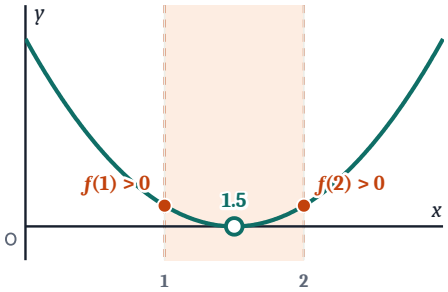
$f(x) = (2x - 3)^2$ from L1 — factorise

$f(1.5) = 0$: there **is** a root, $x = 1.5$. The curve touches the x-axis there and does not cross it.

touches: no change of sign

Sam is wrong: no change of sign does not mean no root (a repeated root, or two roots close together).

same sign ⇏ no root



touches the axis at 1.5: no change of sign

13 $f(x) = e^x - 4x$.
(a) Explain why $x_0 = \ln 4$ cannot be used as a first approximation in the Newton–Raphson method.
(b) Find x_1 when $x_0 = 1.5$ and explain why it is a poor result.
(c) Starting from $x_0 = 2.5$, find x_1, x_2 and x_3 to 4 d.p.

(a) $f'(x) = e^x - 4, f'(\ln 4) = 4 - 4 = 0$ from L4 — $f'(x_0) = 0$: fails

The tangent at $x = \ln 4$ is horizontal: it never meets the x-axis (division by 0).

flat tangent

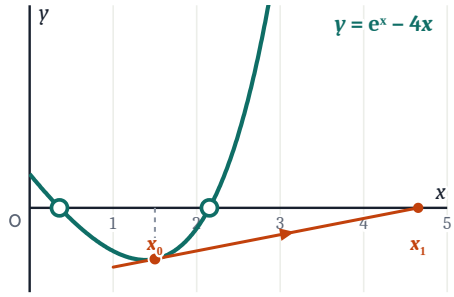
(b) $x_1 = 1.5 - \dfrac{(-1.518\ldots)}{0.4816\ldots} = 4.6521$ — N–R formula

1.5 is close to the turning point at $\ln 4 = 1.386$: the tangent is nearly flat, so x_1 lands far from both roots (0.357 and 2.153).

near a turning point

(c) $x_1 = 2.5 - \dfrac{2.182\ldots}{8.182\ldots} = 2.2333$ — N–R formula

$x_2 = 2.1587, x_3 = 2.1533$ — ANS key: full value



from 1.5: x_1 lands far away

14 Without a calculator. $\int_1^2 \frac{1}{x}\,dx = \ln 2$.

(a) Using 2 strips, find the lower and upper rectangle sums and the trapezium rule estimate, as fractions.

(b) Using the shape of $y = \frac{1}{x}$, give the closest limits that $\ln 2$ must lie between.

(a) $h = ½$: ordinates $1, \frac{2}{3}, \frac{1}{2}$ at $x = 1, \frac{3}{2}, 2$ from L6

h, then the ordinates

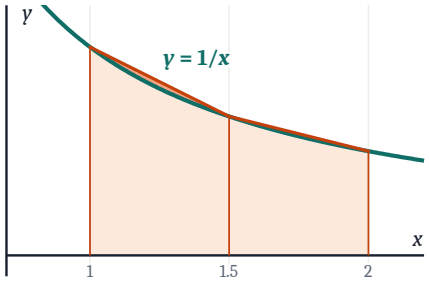
Decreasing: lower (right ends) $½(\frac{2}{3} + \frac{1}{2}) = \frac{7}{12}$; upper (left ends) $½(1 + \frac{2}{3}) = \frac{5}{6}$

decreasing: right ends lower

$T = ¼[1 + 2(\frac{2}{3}) + \frac{1}{2}] = ¼ \times \frac{17}{6} = \frac{17}{24}$ trapezium rule

(b) $y'' = 2x^{-3} > 0$: convex, so T is an overestimate: $\frac{7}{12} < \ln 2 < \frac{17}{24}$

convex: T is an upper limit



convex: T is too big

D · Show that Write every line.

15 The curve $y = e^x - x^3$ has a stationary point P with $0 < x < 1.5$.

(a) Show that the x-coordinate of P satisfies $x = \sqrt{\dfrac{e^x}{3}}$.

(b) Use $x_{n+1} = \sqrt{\dfrac{e^{x_n}}{3}}$ with $x_0 = 1$ to find x_1, x_2 and x_3 to 4 d.p.

(c) Show that the x-coordinate of P is 0.910 correct to 3 d.p.

(a) $\dfrac{dy}{dx} = e^x - 3x^2 = 0$ from L2, L1 — stationary: dy/dx = 0

$3x^2 = e^x \Rightarrow x^2 = \dfrac{e^x}{3} \Rightarrow x = \sqrt{\dfrac{e^x}{3}}$ (as $x > 0$)

rearrange: square root

(b) $x_1 = \sqrt{e^1/3} = 0.9519$ — substitute x_0

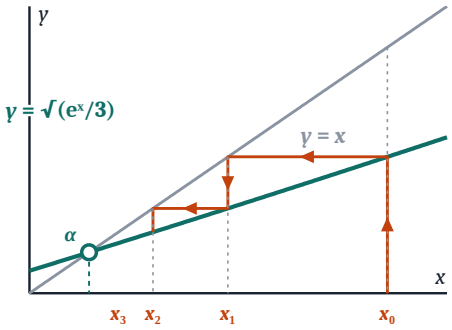
$x_2 = 0.9293$, $x_3 = 0.9188$ — ANS key: full value

(c) With $y' = e^x - 3x^2$: $y'(0.9095) = 0.00151 > 0$, $y'(0.9105) = -0.00147 < 0$

bounds: ±0.0005

Change of sign, y' continuous ⇒ $0.9095 < x < 0.9105$: $x = 0.910$.

continuous ⇒ root



staircase down to 0.910

16 $f(x) = xe^x - 2$.

(a) Show that the Newton–Raphson recurrence for $f(x) = 0$ can be written as $x_{n+1} = \dfrac{x_n^2 + 2e^{-x_n}}{x_n + 1}$.

(b) Use it with $x_0 = 1$ to find x_1, x_2 and x_3 to 4 d.p.

(a) $f'(x) = e^x + xe^x = e^x(x + 1)$ from L4 — product rule

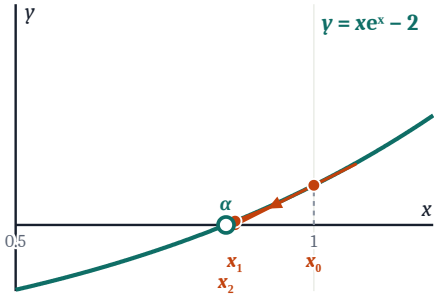
$x_{n+1} = x_n - \dfrac{x_n e^{x_n} - 2}{e^{x_n}(x_n + 1)}$ — N–R formula

$= \dfrac{x_n(x_n + 1)e^{x_n} - x_n e^{x_n} + 2}{e^{x_n}(x_n + 1)} = \dfrac{x_n^2 e^{x_n} + 2}{e^{x_n}(x_n + 1)}$ — one fraction

$= \dfrac{x_n^2 + 2e^{-x_n}}{x_n + 1}$ (÷ e^{x_n} top and bottom) — ÷ e^x top and bottom

(b) $x_1 = \dfrac{1^2 + 2e^{-1}}{1 + 1} = 0.8679$ — substitute x_0

$x_2 = 0.8528$, $x_3 = 0.8526$ — ANS key: full value



two tangents: x_1, x_2

E · Exam-style questions

One AQA-style and one Edexcel-style question. Use the mark scheme to self-assess.

17 *AQA-style.* The curve $y = 3 + 2x - e^x$ crosses the positive x-axis at $x = \alpha$.

(a) Show that α lies between 1.9 and 2.0. [2 marks]

(b) Taking $x_1 = 2$ as a first approximation for α, use the Newton–Raphson method to find x_2, giving your answer to four decimal places. [3 marks]

(c) Explain why $x_1 = \ln 2$ could not be used as a first approximation. [1 mark]

(d) Show that $\alpha = 1.92$ correct to two decimal places. [2 marks]

(a) Let $f(x) = 3 + 2x - e^x$: $f(1.9) = 0.114 > 0$, $f(2) = -0.389 < 0$ from L1, L4 **M1**

substitute both ends

Change of sign, f continuous ⇒ a root in $[1.9, 2]$. **A1** continuous ⇒ root

(b) $f'(x) = 2 - e^x$: $f(2) = -0.3890\ldots$, $f'(2) = -5.389\ldots$ **B1 M1**

differentiate, substitute x_1

$x_2 = 2 - \dfrac{(-0.3890\ldots)}{-5.389\ldots} = 1.9278$ **A1** N–R formula

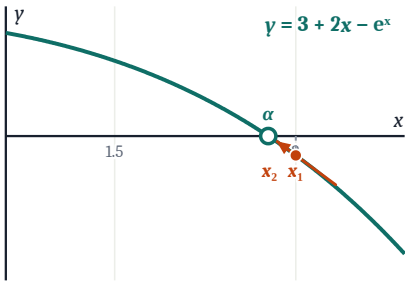
(c) $f'(\ln 2) = 2 - 2 = 0$: a horizontal tangent never meets the x-axis. **E1**

$f'(x_1) = 0$: fails

(d) $f(1.915) = 0.0431 > 0$, $f(1.925) = -0.00515 < 0$ **M1** bounds: ±0.005

Change of sign, f continuous ⇒ $1.915 < \alpha < 1.925$: $\alpha = 1.92$. **A1**

continuous ⇒ root



(b): one tangent from $x_1 = 2$

18 *Edexcel-style.* $f(x) = 4 - x^2 + \ln(x + 1)$, $x > -1$. The equation $f(x) = 0$ has a positive root α.

(a) Show that α lies in $[2.2, 2.3]$. [2 marks]

(b) Show that $f(x) = 0$ can be written as $x = \sqrt{4 + \ln(x + 1)}$. [1 mark]

(c) Using $x_{n+1} = \sqrt{4 + \ln(x_n + 1)}$ with $x_1 = 2$, find x_2, x_3 and x_4 to 4 decimal places. [2 marks]

(d) Use the trapezium rule with 4 strips to estimate $\int_0^2 f(x)\,dx$ to 3 significant figures. [3 marks]

(e) Given that $f''(x) = -2 - 1/(x + 1)^2$, state, with a reason, whether your estimate is an overestimate or an underestimate. [1 mark]

(a) $f(2.2) = 0.323 > 0$, $f(2.3) = -0.0961 < 0$ from L1, L2, L6 **M1**

substitute both ends

Change of sign, f continuous ⇒ $2.2 < \alpha < 2.3$. **A1** continuous ⇒ root

(b) $x^2 = 4 + \ln(x + 1) \Rightarrow x = \sqrt{4 + \ln(x + 1)}$ (as $x > 0$) **B1**

rearrange: square root

(c) $x_2 = \sqrt{4 + \ln(2 + 1)} = 2.2580$ **M1** substitute x_1

$x_3 = 2.2762$, $x_4 = 2.2774$ **A1** ANS key: full value

(d) $h = 0.5$: $y_0 = 4$, $y_1 = 4.1555$, $y_2 = 3.6931$, $y_3 = 2.6663$, $y_4 = 1.0986$ **B1**

h, then the ordinates

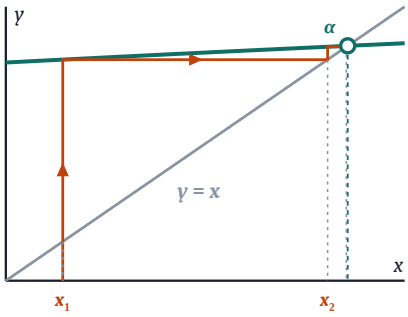
$\approx \tfrac{1}{2} \times 0.5 \times [4 + 2(4.1555 + 3.6931 + 2.6663) + 1.0986]$ **M1**

ends once, middle twice

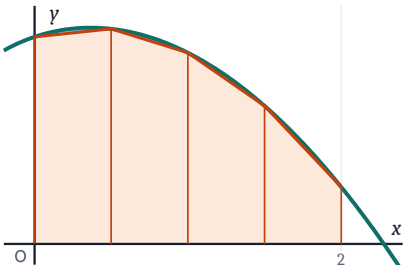
$= 0.25 \times 26.1284 = 6.5321 \approx 6.53$ (3 s.f.) **A1** round at the end

(e) $f''(x) < 0$: concave, so an **under**estimate. **B1**

chords below the curve



(c): x_2, x_3, x_4 close in on α



(d): concave: an underestimate