Trigonometric Identities

Specification: AQA 7357: E5 (proofs E8, A-level) · Edexcel 9MA0: 5.5

INDEPENDENT

You do

A · Simplifying No calculator.

1 Simplify

(a) $3 - 3\sin^2\theta$

(b) $2\sin^2 4\theta + 2\cos^2 4\theta$

(c) $\cos^2\theta - 1$

(d) $\dfrac{\sin\theta}{\tan\theta\cos\theta}$

(a) $3(1-\sin^2\theta) = 3\cos^2\theta$ — factorise; $1-\sin^2 = \cos^2$

(b) $2(\sin^2 4\theta + \cos^2 4\theta) = 2$ — factorise; same angle: = 1

(c) $\cos^2\theta - 1 = -(1-\cos^2\theta) = -\sin^2\theta$ — $-(1-\cos^2\theta)$

(d) $\tan\theta\cos\theta = \dfrac{\sin\theta}{\cos\theta} \times \cos\theta = \sin\theta$, so the fraction

$= \dfrac{\sin\theta}{\sin\theta} = 1$

tan = sin ÷ cos

2 Simplify

(a) $\dfrac{1-\sin^2\theta}{\cos\theta}$

(b) $(\sin\theta + \cos\theta)^2 - 2\sin\theta\cos\theta$

(c) $\sin^4\theta + \sin^2\theta\cos^2\theta$

(d) $7 - 7\cos^2(\tfrac{1}{2}\theta)$

(a) $\dfrac{\cos^2\theta}{\cos\theta} = \cos\theta$ — $1-\sin^2 = \cos^2$; $\cos\theta \neq 0$

(b) $\sin^2\theta + 2\sin\theta\cos\theta + \cos^2\theta - 2\sin\theta\cos\theta$

expand the bracket

$= \sin^2\theta + \cos^2\theta = 1$ — $\sin^2 + \cos^2 = 1$

(c) $\sin^2\theta(\sin^2\theta + \cos^2\theta) = \sin^2\theta$ — factor $\sin^2\theta$

(d) $7(1-\cos^2(\tfrac{1}{2}\theta)) = 7\sin^2(\tfrac{1}{2}\theta)$ — same angle $\tfrac{1}{2}\theta$

3 Write each expression in terms of $\sin\theta$ only.

(a) $3\cos^2\theta - 1$

(b) $\dfrac{\cos^2\theta}{1-\sin\theta}$

(c) $\tan^2\theta\cos^2\theta$

(a) $3(1-\sin^2\theta) - 1 = 2 - 3\sin^2\theta$ — $\cos^2\theta = 1 - \sin^2\theta$

(b) $\dfrac{1-\sin^2\theta}{1-\sin\theta} = \dfrac{(1-\sin\theta)(1+\sin\theta)}{1-\sin\theta}$ — difference of two squares

$= 1 + \sin\theta$ — cancel: $1-\sin\theta \neq 0$

(c) $\dfrac{\sin^2\theta}{\cos^2\theta} \times \cos^2\theta = \sin^2\theta$ — $\tan^2 = \sin^2 \div \cos^2$

4 (a) Given that $3\sin\theta = 7\cos\theta$, find $\tan\theta$.

(b) Given that $2\sin\theta + 5\cos\theta = 0$, find $\tan\theta$.

(c) Given that $\sin x\cos y = 4\cos x\sin y$ ($\cos x \neq 0$, $\cos y \neq 0$), write $\tan x$ in terms of $\tan y$.

(a) If $\cos\theta = 0$ then $\sin\theta = 0$ too: impossible, as $\sin^2\theta + \cos^2\theta = 1$. So $\cos\theta \neq 0$.

check $\cos\theta \neq 0$

$\div 3\cos\theta$: $\tan\theta = \dfrac{7}{3}$ — $\div 3\cos\theta$

(b) $2\sin\theta = -5\cos\theta$; $\cos\theta \neq 0$ (else $\sin\theta = 0$ too)

rearrange; $\cos\theta \neq 0$

$\div 2\cos\theta$: $\tan\theta = -\dfrac{5}{2}$ — $\div 2\cos\theta$

(c) $\div \cos x\cos y$: $\dfrac{\sin x}{\cos x} = 4 \times \dfrac{\sin y}{\cos y}$ — $\div \cos x\cos y$

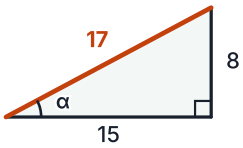
$\tan x = 4\tan y$ — tan = sin ÷ cos

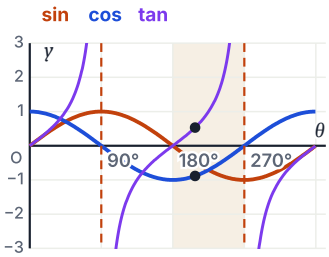
B · One ratio → the others No calculator. Exact answers.

5 $\tan\theta = \frac{8}{15}$ and $180° < \theta < 270°$. Find exactly

(a) $\sin\theta$

(b) $\cos\theta$





draw the triangle and the graphs

Triangle: $\tan\alpha = \frac{8}{15}$ (opposite 8, adjacent 15); hypotenuse = $\sqrt{8^2 + 15^2} = 17$ — Pythagoras: size only

$180° < \theta < 270°$: 180°–270°. There the sine and cosine graphs are below the axis: sin −, cos −. — graph: 180°–270°

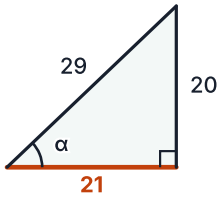
(a) $\sin\theta = -\frac{8}{17}$ — opp ÷ hyp, sign −

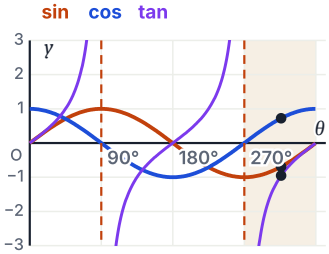
(b) $\cos\theta = -\frac{15}{17}$ — adj ÷ hyp, sign −

6 $\sin\theta = -\frac{20}{29}$ and $270° < \theta < 360°$. Find exactly

(a) $\cos\theta$

(b) $\tan\theta$





draw the triangle and the graphs

Triangle: $\sin\alpha = \frac{20}{29}$ (opposite 20, hypotenuse 29); adjacent = $\sqrt{29^2 - 20^2} = 21$ — Pythagoras: size only

$270° < \theta < 360°$: 270°–360°. There the cosine graph is above the axis and the tan graph is below the axis: cos +, tan −. — graph: 270°–360°

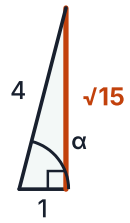
(a) $\cos\theta = \frac{21}{29}$ — adj ÷ hyp, sign +

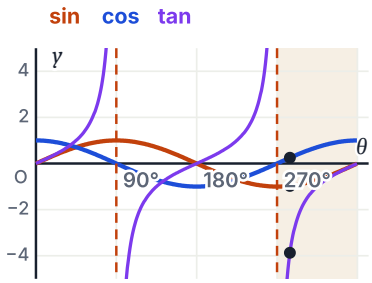
(b) $\tan\theta = -\frac{20}{21}$ — opp ÷ adj, sign −

7 $\cos\theta = \frac{1}{4}$ and θ is reflex. Find exactly

(a) $\sin\theta$

(b) $\tan\theta$





draw the triangle and the graphs

Triangle: $\cos\alpha = \frac{1}{4}$ (adjacent 1, hypotenuse 4); opposite $= \sqrt{4^2 - 1^2} = \sqrt{15}$ Pythagoras: size only

Reflex: 180°–360°; the cosine graph is above the axis there only on 270°–360°. There the sine and tan graphs are below the axis: sin −, tan −.

graph: 270°–360°

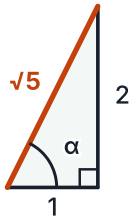
(a) $\sin\theta = -\frac{\sqrt{15}}{4}$ opp ÷ hyp, sign −

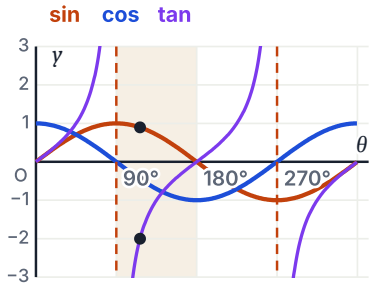
(b) $\tan\theta = -\sqrt{15}$ opp ÷ adj, sign −

8 $\tan\theta = -2$ and θ is obtuse. Find exactly

(a) $\sin\theta$

(b) $\cos\theta$





draw the triangle and the graphs

Triangle: $\tan\alpha = 2$ (opposite 2, adjacent 1); hypotenuse $= \sqrt{2^2 + 1^2} = \sqrt{5}$ Pythagoras: size only

Obtuse: 90°–180°. There the sine graph is above the axis and the cosine graph is below the axis: sin +, cos −.

graph: 90°–180°

(a) $\sin\theta = \frac{2}{\sqrt{5}} = \frac{2\sqrt{5}}{5}$ opp ÷ hyp, rationalise, sign +

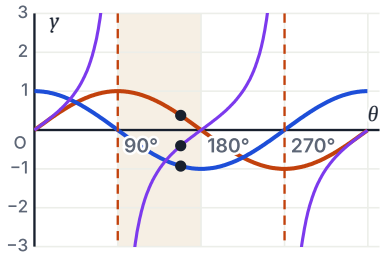
(b) $\cos\theta = -\frac{1}{\sqrt{5}} = -\frac{\sqrt{5}}{5}$ adj ÷ hyp, rationalise, sign −

9 $\sin\theta = \frac{3}{8}$ and θ is obtuse. Using $\sin^2\theta + \cos^2\theta \equiv 1$ (no triangle), find exactly

(a) $\cos\theta$

(b) $\tan\theta$

sin cos tan



sketch the three graphs

$\cos^2\theta = 1 - \sin^2\theta = 1 - (\frac{3}{8})^2 = \frac{55}{64}$ use $\sin^2 + \cos^2 \equiv 1$

$\cos\theta = \pm\frac{\sqrt{55}}{8}$ square root: ±

Obtuse: 90°–180°. There the cosine and tan graphs are below the axis: cos −, tan −.

graph: 90°–180°

(a) $\cos\theta = -\frac{\sqrt{55}}{8}$ take the sign: −

(b) $\tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{3}{8} \div (-\frac{\sqrt{55}}{8}) = -\frac{3}{\sqrt{55}} = -\frac{3\sqrt{55}}{55}$

tan = sin ÷ cos, rationalise

C · Reasoning No calculator for this section.

10 $\sin\theta = 2\cos\theta$.

(a) Explain why $\cos\theta$ cannot be 0.

(b) Hence find $\tan\theta$.

(a) Suppose $\cos\theta = 0$. Then $\sin\theta = 2 \times 0 = 0$ as well.

suppose cos θ = 0

But $\sin^2\theta + \cos^2\theta = 1$ and $0 + 0 \neq 1$: a contradiction. So $\cos\theta \neq 0$.

contradiction

(b) ÷ $\cos\theta$ (allowed: $\cos\theta \neq 0$): $\tan\theta = 2$ ÷ cos θ (cos θ ≠ 0)

11 True or false? Give a reason or a counter-example.

(a) $\sin\theta + \cos\theta \equiv 1$

(b) $1 - \sin^2\theta \equiv \cos^2\theta$

(c) $\sqrt{1 - \cos^2\theta} \equiv \sin\theta$

(a) False: θ = 45° gives $\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2} = \sqrt{2}$, not 1.

one counter-example

(b) True: it is $\sin^2\theta + \cos^2\theta \equiv 1$ rearranged.

rearranged identity

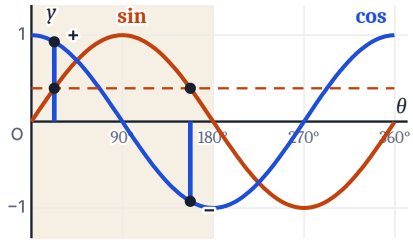
(c) False: θ = 270° gives $\sqrt{1 - 0} = 1$, but $\sin 270° = -1$. ($\sqrt{1 - \cos^2\theta} = |\sin\theta|$.)

√(sin²θ) = |sin θ|

12 Given that $\sin\theta = \frac{5}{13}$ and $0° \le \theta < 360°$:

(a) find the two possible values of $\cos\theta$

(b) state the quadrant of θ for each value.



sketch the graphs

(a) $\cos^2\theta = 1 - (\frac{5}{13})^2 = \frac{144}{169}$ — use $\sin^2 + \cos^2 = 1$

$\cos\theta = \pm\frac{12}{13}$ — square root: ±

(b) The sine graph is above the axis between 0° and 180°. There the cosine graph is above the axis up to 90°, below it from 90° to 180°.

graphs: 0°–180°

$\cos\theta = \frac{12}{13}$: quadrant 1; $\cos\theta = -\frac{12}{13}$: quadrant 2.

match the signs

D · Proofs and eliminating θ

No calculator. Full proofs: A-level for AQA (E8), AS for Edexcel (5.5).

13 Prove that $(1 - \cos\theta)(1 + \cos\theta) \equiv \tan^2\theta\cos^2\theta$.

LHS $\equiv 1 - \cos^2\theta$ — expand: difference of squares

$\equiv \sin^2\theta$ — $1 - \cos^2\theta \equiv \sin^2\theta$

$\equiv \frac{\sin^2\theta}{\cos^2\theta} \times \cos^2\theta$ — × and ÷ $\cos^2\theta$ ($\neq 0$: tan θ defined)

$\equiv \tan^2\theta\cos^2\theta \equiv$ RHS — $\sin^2 \div \cos^2 = \tan^2$

14 Prove that $\frac{1}{\cos\theta} - \sin\theta\tan\theta \equiv \cos\theta$.

LHS $\equiv \frac{1}{\cos\theta} - \sin\theta \times \frac{\sin\theta}{\cos\theta}$ — tan = sin ÷ cos

$\equiv \frac{1 - \sin^2\theta}{\cos\theta}$ — one fraction

$\equiv \frac{\cos^2\theta}{\cos\theta}$ — $1 - \sin^2\theta \equiv \cos^2\theta$

$\equiv \cos\theta \equiv$ RHS — cancel: $\cos\theta \neq 0$ (denominator)

15 Eliminate θ to find an equation connecting x and y.

(a) $x = 2\sin\theta$, $y = 5\cos\theta$

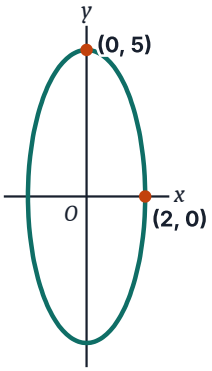
(b) $x = 3 + \cos\theta$, $y = \sin\theta - 1$

(a) $\sin\theta = \frac{x}{2}$, $\cos\theta = \frac{y}{5}$ — make sin θ, cos θ the subject

$\frac{x^2}{4} + \frac{y^2}{25} = 1$, so $25x^2 + 4y^2 = 100$ — sin² + cos² = 1; × 100

(b) $\cos\theta = x - 3$, $\sin\theta = y + 1$ — make cos θ, sin θ the subject

$(x-3)^2 + (y+1)^2 = 1$: a circle, centre $(3, -1)$, radius 1 — sin² + cos² = 1





(b) a circle

(a) $25x^2 + 4y^2 = 100$

(a) an ellipse; (b) the circle centre (3, −1), radius 1

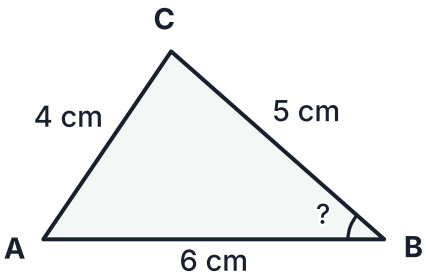
E · Exam-style question No calculator. Use the mark scheme to self-assess.

16 Triangle ABC has $AB = 6$ cm, $BC = 5$ cm and $AC = 4$ cm.

(a) Show that $\cos B = \frac{3}{4}$. [3 marks]

(b) Hence find the exact value of $\sin B$. [2 marks]

(c) Find the exact area of the triangle. [2 marks]



draw a diagram first

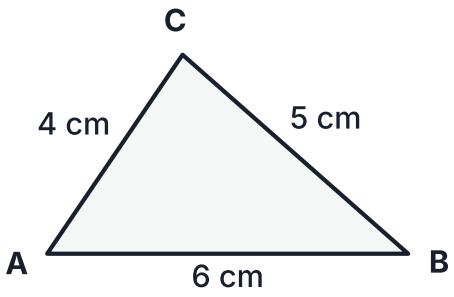
(a) $AC^2 = AB^2 + BC^2 - 2 \times AB \times BC \times \cos B$ **M1** — cosine rule: AC opposite B

$16 = 36 + 25 - 60\cos B$, so $\cos B = \frac{45}{60} = \frac{3}{4}$ **A1 A1** — substitute; ÷ 60

(b) $\sin^2 B = 1 - (\frac{3}{4})^2 = \frac{7}{16}$ **M1** — $\sin^2 = 1 - \cos^2$

B is between 0° and 180°, where the sine graph is above the axis: $\sin B = \frac{\sqrt{7}}{4}$ **A1** — graph: above, 0°–180°

(c) Area $= \frac{1}{2} \times 6 \times 5 \times \sin B = 15 \times \frac{\sqrt{7}}{4} = \frac{15\sqrt{7}}{4}$ cm² **M1 A1** — ½ac sin B



cos B = ¾, sin B = ¼√7, area $\frac{15}{4}\sqrt{7}$ cm²

cos B = ¾, sin B = ¼√7