Using the Addition Formulae

Specification: AQA 7357: E3, E6 · Edexcel 9MA0: 5.3, 5.6

INDEPENDENT

You do

A · Exact values No calculator.

1 Find the exact value of:

(a) $\cos 75°$

(b) $\cos 105°$

(c) $\sin 165°$

(d) $\tan 75°$

(a) $\cos 45° \cos 30° - \sin 45° \sin 30° = \frac{\sqrt{2}}{2} \times \frac{\sqrt{3}}{2} - \frac{\sqrt{2}}{2} \times \frac{1}{2} = \frac{\sqrt{6} - \sqrt{2}}{4}$

split: 45° + 30°; cos(A + B)

(b) $\cos 60° \cos 45° - \sin 60° \sin 45° = \frac{1}{2} \times \frac{\sqrt{2}}{2} - \frac{\sqrt{3}}{2} \times \frac{\sqrt{2}}{2} = \frac{\sqrt{2} - \sqrt{6}}{4}$

split: 60° + 45°; cos(A + B)

(c)

$\sin 120° \cos 45° + \cos 120° \sin 45° = \frac{\sqrt{3}}{2} \times \frac{\sqrt{2}}{2} + (-\frac{1}{2}) \times \frac{\sqrt{2}}{2} = \frac{\sqrt{6} - \sqrt{2}}{4}$

split: 120° + 45°; sin(A + B)

(d) $\tan(45° + 30°) = \frac{1 + \frac{1}{\sqrt{3}}}{1 - \frac{1}{\sqrt{3}}} = \frac{\sqrt{3} + 1}{\sqrt{3} - 1}$

tan(A + B); × √3 top and bottom

$= \frac{(\sqrt{3} + 1)^2}{3 - 1} = \frac{4 + 2\sqrt{3}}{2} = 2 + \sqrt{3}$

rationalise

2 Find the exact value of:

(a) $\cos \frac{\pi}{12}$

(b) $\sin \frac{5\pi}{12}$

(c) $\cos \frac{7\pi}{12}$

(d) $\tan \frac{\pi}{12}$

(a)

$\cos(\frac{\pi}{3} - \frac{\pi}{4}) = \frac{1}{2} \times \frac{\sqrt{2}}{2} + \frac{\sqrt{3}}{2} \times \frac{\sqrt{2}}{2} = \frac{\sqrt{6} + \sqrt{2}}{4}$

split: π/3 − π/4; cos(A − B)

(b)

$\sin(\frac{\pi}{4} + \frac{\pi}{6}) = \frac{\sqrt{2}}{2} \times \frac{\sqrt{3}}{2} + \frac{\sqrt{2}}{2} \times \frac{1}{2} = \frac{\sqrt{6} + \sqrt{2}}{4}$

split: π/4 + π/6; sin(A + B)

(c)

$\cos(\frac{\pi}{3} + \frac{\pi}{4}) = \frac{1}{2} \times \frac{\sqrt{2}}{2} - \frac{\sqrt{3}}{2} \times \frac{\sqrt{2}}{2} = \frac{\sqrt{2} - \sqrt{6}}{4}$

split: π/3 + π/4; cos(A + B)

(d) $\tan(\frac{\pi}{4} - \frac{\pi}{6}) = \frac{1 - \frac{1}{\sqrt{3}}}{1 + \frac{1}{\sqrt{3}}} = \frac{\sqrt{3} - 1}{\sqrt{3} + 1}$

tan(A − B); × √3 top and bottom

$= \frac{(\sqrt{3} - 1)^2}{3 - 1} = \frac{4 - 2\sqrt{3}}{2} = 2 - \sqrt{3}$

rationalise

3 Find the exact values:

(a) $\sin A = \frac{5}{13}$, A acute. Find $\cos A$ and $\tan A$.

(b) $\cos B = -\frac{3}{5}$, B obtuse. Find $\sin B$ and $\tan B$.

(c) $\tan C = \frac{7}{24}$, $180° < C < 270°$. Find $\sin C$ and $\cos C$.

(a) $\cos^2 A = 1 - \frac{25}{169} = \frac{144}{169} \Rightarrow \cos A = \frac{12}{13}$

acute: cos +

$\tan A = \frac{\sin A}{\cos A} = \frac{5}{12}$

tan = sin ÷ cos

(b) $\sin^2 B = 1 - \frac{9}{25} = \frac{16}{25} \Rightarrow \sin B = \frac{4}{5}$

obtuse: 2nd quadrant, sin +

$\tan B = \frac{\frac{4}{5}}{-\frac{3}{5}} = -\frac{4}{3}$

tan = sin ÷ cos

(c) Hypotenuse $\sqrt{7^2 + 24^2} = 25$: triangle 7, 24, 25

right-angled triangle

$\sin C = -\frac{7}{25}, \cos C = -\frac{24}{25}$

3rd quadrant: sin −, cos −

B · Substituting Exact answers. No calculator.

4 $\sin A = \frac{3}{5}$ and $\cos B = \frac{5}{13}$, where A and B are acute.

Find:

(a) $\sin(A + B)$

(b) $\cos(A + B)$

(c) $\tan(A - B)$

$\cos A = \frac{4}{5}, \sin B = \frac{12}{13}$ — acute: both +; triangles 3, 4, 5 and 5, 12, 13

(a) $\sin A \cos B + \cos A \sin B = \frac{3}{5} \times \frac{5}{13} + \frac{4}{5} \times \frac{12}{13} = \frac{63}{65}$ — sin(A + B)

(b) $\cos A \cos B - \sin A \sin B = \frac{4}{5} \times \frac{5}{13} - \frac{3}{5} \times \frac{12}{13} = -\frac{16}{65}$ — cos(A + B)

(c) $\tan A = \frac{3}{4}, \tan B = \frac{12}{5}$ — tan = sin ÷ cos

$\frac{\tan A - \tan B}{1 + \tan A \tan B} = \frac{\frac{3}{4} - \frac{12}{5}}{1 + \frac{36}{20}} = \frac{-\frac{33}{20}}{\frac{56}{20}} = -\frac{33}{56}$ — tan(A − B)

5 $\sin A = -\frac{4}{5}$ with $180° < A < 270°$, and $\cos B = -\frac{7}{25}$ with B obtuse. Find:

(a) $\sin(A - B)$

(b) $\cos(A + B)$

$\cos^2 A = 1 - \frac{16}{25} = \frac{9}{25} \Rightarrow \cos A = -\frac{3}{5}$ — 3rd quadrant: cos −

$\sin^2 B = 1 - \frac{49}{625} = \frac{576}{625} \Rightarrow \sin B = \frac{24}{25}$ — 2nd quadrant: sin +

(a) $\sin A \cos B - \cos A \sin B = (-\frac{4}{5})(-\frac{7}{25}) - (-\frac{3}{5})(\frac{24}{25})$ — sin(A − B)

$= \frac{28}{125} + \frac{72}{125} = \frac{100}{125} = \frac{4}{5}$ — simplify

(b) $\cos A \cos B - \sin A \sin B = (-\frac{3}{5})(-\frac{7}{25}) - (-\frac{4}{5})(\frac{24}{25})$ — cos(A + B)

$= \frac{21}{125} + \frac{96}{125} = \frac{117}{125}$ — add

6 $\tan A = \frac{1}{2}$ and $\tan B = \frac{1}{3}$, where A and B are acute.

Show that $A + B = 45°$.

$\tan(A + B) = \frac{\frac{1}{2} + \frac{1}{3}}{1 - \frac{1}{6}} = \frac{\frac{5}{6}}{\frac{5}{6}} = 1$ — tan(A + B)

$0° < A + B < 180°$ and $\tan(A + B) = 1$, so $A + B = 45°$. — find the angle

C · Reasoning No calculator.

7 $\tan A = 2$ and $\tan B = 3$, where A and B are acute. Find $A + B$. Explain why it is not 45°.

$\tan(A + B) = \frac{2 + 3}{1 - 6} = -1$ — tan(A + B)

Both $A, B > 45°$ ($\tan > 1$), so $90° < A + B < 180°$. — estimate the range

$A + B = 135°$ — tan = −1 in 2nd quadrant

8 Show that $\sec 15° = \sqrt{6} - \sqrt{2}$.

$\sec 15° = \frac{1}{\cos 15°} = \frac{4}{\sqrt{6} + \sqrt{2}}$ — Example 1

$= \frac{4(\sqrt{6} - \sqrt{2})}{6 - 2}$ — rationalise

$= \sqrt{6} - \sqrt{2}$ — simplify

9 Show that $\cos 75° + \cos 15° = \frac{\sqrt{6}}{2}$.

$\frac{\sqrt{6} - \sqrt{2}}{4} + \frac{\sqrt{6} + \sqrt{2}}{4}$ — exact values

$= \frac{2\sqrt{6}}{4} = \frac{\sqrt{6}}{2}$ — add

D · Algebra

10 Given that $\sin(x + y) = 3\cos(x + y)$, express $\tan x$ in terms of $\tan y$.

$\sin x \cos y + \cos x \sin y = 3 \cos x \cos y - 3 \sin x \sin y$ — expand both sides

$\tan x + \tan y = 3 - 3 \tan x \tan y$ — ÷ cos x cos y

$\tan x (1 + 3 \tan y) = 3 - \tan y$ — factorise

$\tan x = \frac{3 - \tan y}{1 + 3 \tan y}$ — ÷ (1 + 3 tan y)

11 Given that $2\sin(x - y) = \cos(x + y)$, express $\tan x$ in terms of $\tan y$.

$2 \tan x - 2 \tan y = 1 - \tan x \tan y$ — expand, ÷ cos x cos y

$\tan x (2 + \tan y) = 1 + 2 \tan y$ — factorise

$\tan x = \frac{1 + 2 \tan y}{2 + \tan y}$ — ÷ (2 + tan y)

12 Given that $\tan(x + 45°) = 3$, find the exact value of $\tan x$.

$\dfrac{\tan x + 1}{1 - \tan x} = 3$ — tan(A + B); tan 45° = 1

$\tan x + 1 = 3 - 3\tan x$ — × (1 − tan x)

$\tan x = \dfrac{1}{2}$ — solve

13 Given that $\tan(x - 60°) = 2$, show that $\tan x = -\dfrac{8 + 5\sqrt{3}}{11}$.

$\dfrac{\tan x - \sqrt{3}}{1 + \sqrt{3}\tan x} = 2$ — tan(A − B); tan 60° = √3

$\tan x - \sqrt{3} = 2 + 2\sqrt{3}\tan x$ — × denominator

$\tan x(1 - 2\sqrt{3}) = 2 + \sqrt{3}$ — factorise

$\tan x = \dfrac{(2 + \sqrt{3})(1 + 2\sqrt{3})}{1 - 12} = -\dfrac{8 + 5\sqrt{3}}{11}$ — rationalise

E · Exam-style question AQA-style. Use the mark scheme to self-assess.

14 (a) Use an appropriate addition formula to show that $\cos 15° = \dfrac{\sqrt{6} + \sqrt{2}}{4}$. [3 marks]

(b) Hence find the exact value of $\sec^2 15°$, giving your answer in the form $a + b\sqrt{3}$, where a and b are integers. [3 marks]

(a) $\cos(45° - 30°) = \cos 45° \cos 30° + \sin 45° \sin 30°$ **M1** — split: 45° − 30°

$= \dfrac{\sqrt{2}}{2} \times \dfrac{\sqrt{3}}{2} + \dfrac{\sqrt{2}}{2} \times \dfrac{1}{2}$ **M1** — exact values

$= \dfrac{\sqrt{6} + \sqrt{2}}{4}$ **A1** — simplify surds

(b) $\cos^2 15° = \dfrac{6 + 2 + 2\sqrt{12}}{16} = \dfrac{2 + \sqrt{3}}{4}$ **M1** — square

$\sec^2 15° = \dfrac{4}{2 + \sqrt{3}} = \dfrac{4(2 - \sqrt{3})}{4 - 3}$ **M1** — rationalise

$= 8 - 4\sqrt{3}$ **A1** — simplify