Writing a cos x + b sin x as R cos(x – α)

Specification: AQA 7357: E6, E3 · Edexcel 9MA0: 5.6, 5.3

INDEPENDENT

You do

A · Writing in R-form R > 0. Give R exactly and α (0° < α < 90°) to 1 decimal place.

1 (a) $6\cos x + 8\sin x \equiv R\cos(x - \alpha)$
(b) $2\sin x + 3\cos x \equiv R\sin(x + \alpha)$
(c) $3\sin x - 2\cos x \equiv R\sin(x - \alpha)$
(d) $4\cos x - 7\sin x \equiv R\cos(x + \alpha)$

(a) $R\cos\alpha = 6, R\sin\alpha = 8 \Rightarrow R = \sqrt{36 + 64} = 10$
compare coefficients; square and add

$\tan\alpha = \frac{8}{6} \Rightarrow \alpha = 53.1°$: $10\cos(x - 53.1°)$ divide: tan α

(b) $R\cos\alpha = 2, R\sin\alpha = 3 \Rightarrow R = \sqrt{4 + 9} = \sqrt{13}$
compare coefficients; square and add

$\tan\alpha = \frac{3}{2} \Rightarrow \alpha = 56.3°$: $\sqrt{13}\sin(x + 56.3°)$ divide: tan α

(c) $R\cos\alpha = 3, R\sin\alpha = 2 \Rightarrow R = \sqrt{9 + 4} = \sqrt{13}$
compare coefficients; square and add

$\tan\alpha = \frac{2}{3} \Rightarrow \alpha = 33.7°$: $\sqrt{13}\sin(x - 33.7°)$ divide: tan α

(d) $R\cos\alpha = 4, R\sin\alpha = 7 \Rightarrow R = \sqrt{16 + 49} = \sqrt{65}$
compare coefficients; square and add

$\tan\alpha = \frac{7}{4} \Rightarrow \alpha = 60.3°$: $\sqrt{65}\cos(x + 60.3°)$ divide: tan α

2 Give exact values of R and α, with $0 < \alpha < \frac{\pi}{2}$:
(a) $\sin x + \cos x \equiv R\sin(x + \alpha)$
(b) $\sqrt{3}\cos x + \sin x \equiv R\cos(x - \alpha)$
(c) $\cos x - \sin x \equiv R\cos(x + \alpha)$

(a) $R\cos\alpha = 1, R\sin\alpha = 1$: $R = \sqrt{2}, \tan\alpha = 1 \Rightarrow \alpha = \frac{\pi}{4}$:
$\sqrt{2}\sin(x + \frac{\pi}{4})$
compare; square and add; divide

(b) $R\cos\alpha = \sqrt{3}, R\sin\alpha = 1$:
$R = \sqrt{3 + 1} = 2, \tan\alpha = \frac{1}{\sqrt{3}} \Rightarrow \alpha = \frac{\pi}{6}$: $2\cos(x - \frac{\pi}{6})$
compare; square and add; divide

(c) $R\cos\alpha = 1, R\sin\alpha = 1$: $R = \sqrt{2}, \alpha = \frac{\pi}{4}$: $\sqrt{2}\cos(x + \frac{\pi}{4})$
minus in the middle: cos(x + α)

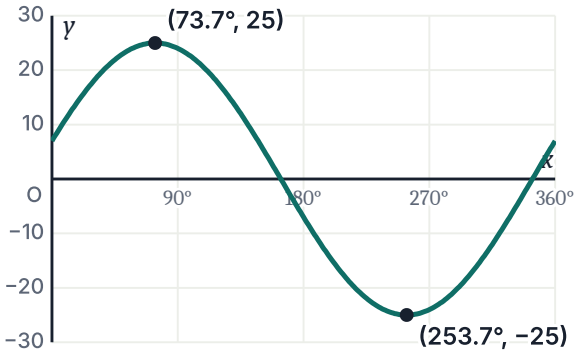
B · Maximum and minimum Use the R-form first.

3 (a) Write $24\sin x + 7\cos x$ in the form $R\sin(x + \alpha)$.
(b) State its maximum value and the value of x in $0° \le x < 360°$ where it occurs.

(a) $R\cos\alpha = 24, R\sin\alpha = 7 \Rightarrow R = \sqrt{576 + 49} = 25$
compare coefficients; square and add

$\tan\alpha = \frac{7}{24} \Rightarrow \alpha = 16.3°$: $25\sin(x + 16.3°)$ divide: tan α

(b) Max 25 when $x + 16.3° = 90°$: $x = 73.7°$ max: bracket = 90°



sketch: max and min

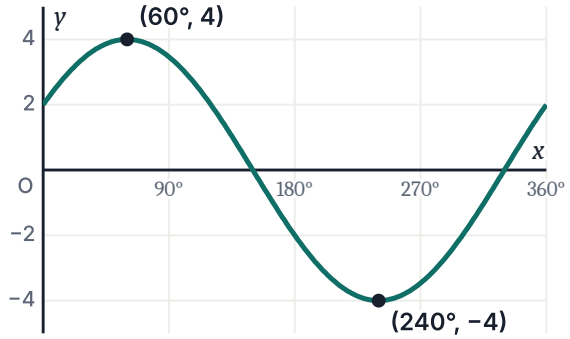
4 (a) Write $f(x) = 2\cos x + 2\sqrt{3}\sin x$ in the form $R\cos(x - \alpha)$.
(b) Find the maximum and minimum of $f(x)$ and where they occur in $0° \le x < 360°$.

(a) $R\cos\alpha = 2, R\sin\alpha = 2\sqrt{3} \Rightarrow R = \sqrt{4 + 12} = 4$
compare coefficients; square and add

$\tan\alpha = \sqrt{3} \Rightarrow \alpha = 60°$: $f(x) = 4\cos(x - 60°)$ divide: tan α

(b) Max 4 when $x - 60° = 0°$: $x = 60°$ max: bracket = 0°

Min −4 when $x - 60° = 180°$: $x = 240°$ min: bracket = 180°



sketch: max and min

5 Find the maximum and minimum values of:
(a) $g(x) = 10 + 3\cos x + 4\sin x$

(b) $h(x) = \dfrac{1}{7 + 3\cos x - 4\sin x}$

(a) $3\cos x + 4\sin x = 5\cos(x - 53.1°)$, $R = \sqrt{9 + 16} = 5$

R-form first

It lies between -5 and 5: max $10 + 5 = 15$, min $10 - 5 = 5$

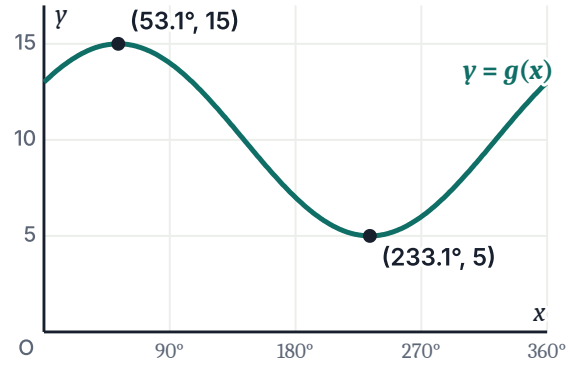
range: $-R$ to R

(b) $3\cos x - 4\sin x = 5\cos(x + 53.1°)$, so the denominator lies between 2 and 12.

R-form; range of denominator

Max of h is $\frac{1}{2}$; min is $\frac{1}{12}$

small denominator, big fraction



sketch of g

C · Reasoning

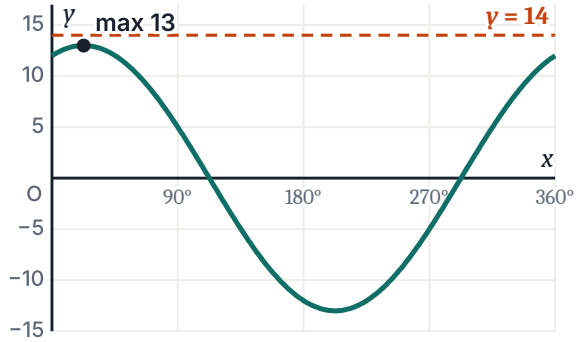
6 Explain why the equation $5\sin x + 12\cos x = 14$ has no solutions.

$5\sin x + 12\cos x = 13\sin(x + 67.4°)$

R-form first

Its largest value is 13, so it can never equal 14.

max is R



line above the maximum

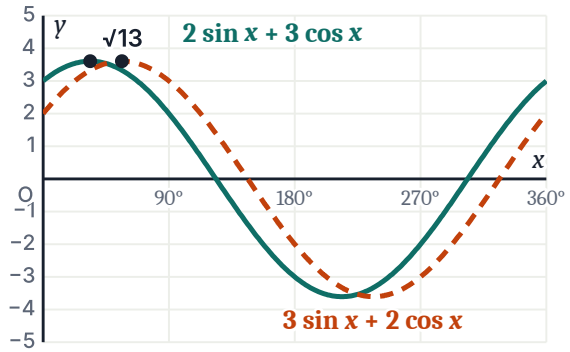
7 Without working out α, which has the larger maximum: $2\sin x + 3\cos x$ or $3\sin x + 2\cos x$?

Both have $R = \sqrt{4 + 9} = \sqrt{13}$.

R = √(a² + b²)

They have the same maximum, $\sqrt{13}$.

same R



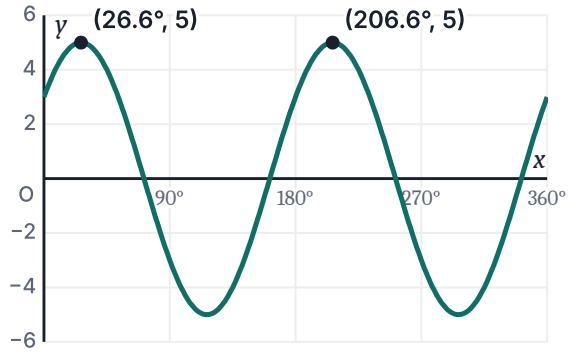
same height, different place

8 $3\cos 2x + 4\sin 2x$: state its maximum value and how many times it reaches it for $0° \le x < 360°$.

$R = 5$, so the maximum is 5. — max is R

The angle is $2x$: period 180°, so two complete waves. — period 180°: 2 waves

It reaches 5 twice. — one max per wave



two maxima

D · Sketching

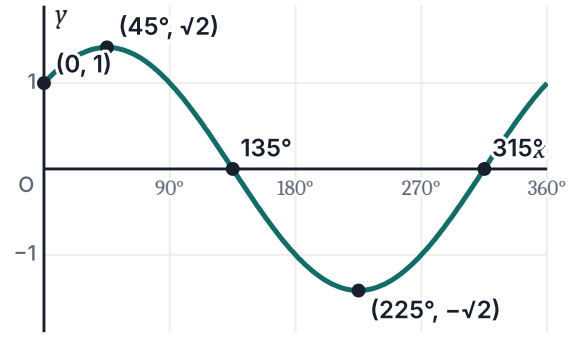
9 (a) Write $\sin x + \cos x$ in the form $R\sin(x + \alpha)$.
(b) Sketch $y = \sin x + \cos x$ for $0° \le x \le 360°$. Label the turning points and the points where it meets the axes.

(a) $R\cos\alpha = 1, R\sin\alpha = 1$: $R = \sqrt{2}$, $\tan\alpha = 1 \Rightarrow \alpha = 45°$: $\sqrt{2}\sin(x + 45°)$ — compare; square and add; divide

(b) $x = 0$: $y = \sqrt{2}\sin 45° = 1$ — y-intercept: put x = 0

$x + 45° = 180°, 360° \Rightarrow x = 135°, 315°$ — x-intercepts: sin = 0

Max $\sqrt{2}$ when $x + 45° = 90°$: $(45°, \sqrt{2})$; min when $x + 45° = 270°$: $(225°, -\sqrt{2})$ — turning points



sketch

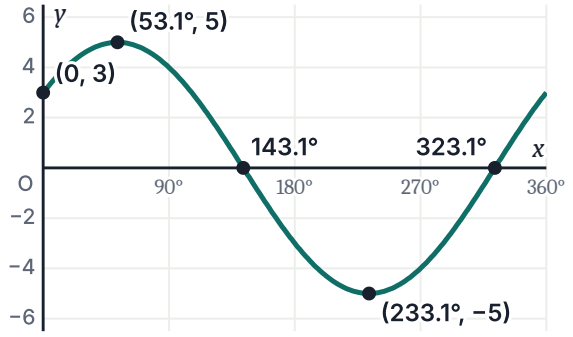
10 Sketch $y = 3\cos x + 4\sin x$ for $0° \le x \le 360°$. Label the turning points and the points where it meets the axes.

$y = 5\cos(x - 53.1°)$ — Explore

$x = 0$: $y = 3\cos 0 + 4\sin 0 = 3$ — y-intercept: put x = 0

$x - 53.1° = 90°, 270° \Rightarrow x = 143.1°, 323.1°$ — x-intercepts: cos = 0

Max 5 when $x - 53.1° = 0°$: $(53.1°, 5)$; min when $x - 53.1° = 180°$: $(233.1°, -5)$ — turning points



sketch

E · Exam-style question AQA-style. Use the mark scheme to self-assess.

11 (a) Express $6\sin x + 2.5\cos x$ in the form $R\sin(x+\alpha)$, where $R > 0$ and $0° < \alpha < 90°$. Give α to 1 decimal place. [3 marks]

(b) State the maximum value of $6\sin x + 2.5\cos x$ and the smallest positive value of x at which it occurs. [2 marks]

(c) Find the minimum value of $\dfrac{1}{8 - 6\sin x - 2.5\cos x}$. [2 marks]

(a) $R\cos\alpha = 6, R\sin\alpha = 2.5$ **M1** — compare coefficients

$R = \sqrt{36 + 6.25} = 6.5$ **A1** — square and add

$\tan\alpha = \dfrac{2.5}{6} \Rightarrow \alpha = 22.6°$ **A1** — divide: tan α

(b) Max 6.5 **B1** — max is R

$x + 22.6° = 90° \Rightarrow x = 67.4°$ **B1** — max: bracket = 90°

(c) $8 - 6.5\sin(x + 22.6°)$ has largest value $8 + 6.5 = 14.5$ **M1** — largest denominator

Minimum $= \dfrac{1}{14.5} = \dfrac{2}{29}$ **A1** — small fraction