# Solving Equations Using R-Form

Specification: AQA 7357: E6, E7 · Edexcel 9MA0: 5.6, 5.7

INDEPENDENT

## You do

**A · Solving** Solve for $0° \le x < 360°$ unless stated. Give answers to 1 decimal place.

**1** Use $3\cos x + 4\sin x \equiv 5\cos(x - 53.13°)$ to solve:

(a) $3\cos x + 4\sin x = 1$

(b) $3\cos x + 4\sin x = -2.5$

(c) $3\cos x + 4\sin x = 5$

(a) $\cos(x - 53.13°) = 0.2$: $x - 53.13° = 78.46°, 281.54°$ ÷ 5; calculator

$x = 131.6°, 334.7°$ add α

(b) $\cos(x - 53.13°) = -0.5$: $x - 53.13° = 120°, 240°$ ÷ 5; CAST

$x = 173.1°, 293.1°$ add α

(c) $\cos(x - 53.13°) = 1$: $x = 53.1°$ cos = 1: one solution

**2** (a) $2\sin x + 3\cos x = 1$

(b) $5\cos x - 2\sin x = 4$

(a) $R\cos\alpha = 2$, $R\sin\alpha = 3$: $R = \sqrt{13}$, $\tan\alpha = \frac{3}{2} \Rightarrow \alpha = 56.31°$

R-form: R sin(x + α)

$\sqrt{13}\sin(x + 56.31°) = 1 \Rightarrow \sin(x + 56.31°) = 0.2774$ ÷ R

$56.31° \le x + 56.31° < 416.31°$: $16.10°$ is outside, so

$x + 56.31° = 163.90°, 376.10°$

shift the interval; 180° − x, + 360°

$x = 107.6°, 319.8°$ subtract α

(b) $R\cos\alpha = 5$, $R\sin\alpha = 2$:

$R = \sqrt{29}$, $\tan\alpha = \frac{2}{5} \Rightarrow \alpha = 21.80°$

R-form: minus gives cos(x + α)

$\sqrt{29}\cos(x + 21.80°) = 4 \Rightarrow \cos(x + 21.80°) = 0.7428$ ÷ R

$21.80° \le x + 21.80° < 381.80°$: $x + 21.80° = 42.03°, 317.97°$

shift the interval; 360° − x

$x = 20.2°, 296.2°$ subtract α

**3** Solve for $0 \le x < 2\pi$. Give exact answers.

(a) $\sin x + \cos x = \frac{1}{\sqrt{2}}$

(b) $\sqrt{3}\cos x + \sin x = 1$

(a) $R = \sqrt{2}$, $\alpha = \frac{\pi}{4}$:

$\sqrt{2}\sin(x + \frac{\pi}{4}) = \frac{1}{\sqrt{2}} \Rightarrow \sin(x + \frac{\pi}{4}) = \frac{1}{2}$

R-form; ÷ R

$\frac{\pi}{4} \le x + \frac{\pi}{4} < \frac{9\pi}{4}$: $\frac{\pi}{6}$ is outside, so $x + \frac{\pi}{4} = \frac{5\pi}{6}, \frac{13\pi}{6}$

shift the interval

$x = \frac{7\pi}{12}, \frac{23\pi}{12}$ subtract π/4

(b) $R = \sqrt{3+1} = 2$, $\alpha = \frac{\pi}{6}$:

$2\cos(x - \frac{\pi}{6}) = 1 \Rightarrow \cos(x - \frac{\pi}{6}) = \frac{1}{2}$

R-form; ÷ R

$-\frac{\pi}{6} \le x - \frac{\pi}{6} < \frac{11\pi}{6}$: $x - \frac{\pi}{6} = \frac{\pi}{3}, \frac{5\pi}{3}$ shift the interval

$x = \frac{\pi}{2}, \frac{11\pi}{6}$ add π/6

## B · Maximum and minimum

**4** $f(x) = \dfrac{6}{4 + \sin x + \sqrt{3}\cos x}$, $0 \le x < 2\pi$. Find the maximum and minimum values of $f(x)$ and where they occur.

$R = \sqrt{1+3} = 2, \tan\alpha = \sqrt{3} \Rightarrow \alpha = \frac{\pi}{3}$:

$\sin x + \sqrt{3}\cos x = 2\sin(x + \frac{\pi}{3})$

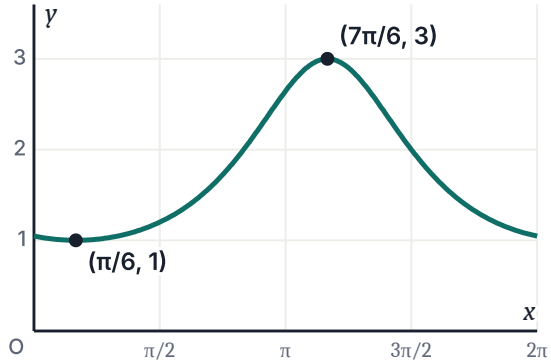
R-form first

Denominator $4 + 2\sin(x + \frac{\pi}{3})$ lies between 2 and 6.

range of the denominator

Max 3 when $x + \frac{\pi}{3} = \frac{3\pi}{2}$: $x = \frac{7\pi}{6}$ — smallest denominator: max

Min 1 when $x + \frac{\pi}{3} = \frac{\pi}{2}$: $x = \frac{\pi}{6}$ — largest denominator: min




sketch: max and min

**5** Find the maximum and minimum values of:
(a) $(3\cos x + 4\sin x)^2$
(b) $5 + 2(5\sin x - 12\cos x)$

(a) $3\cos x + 4\sin x$ lies between −5 and 5, so its square lies between 0 and 25.

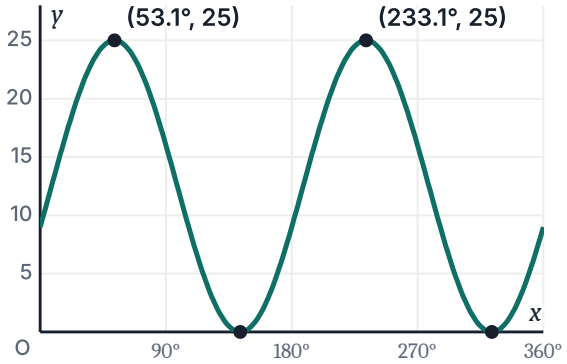
range: −R to R

Max 25, min 0 — square: smallest is 0

(b) $5\sin x - 12\cos x = 13\sin(x - 67.38°)$,
$R = \sqrt{25 + 144} = 13$: it lies between −13 and 13.

R-form; range: −R to R

Max $5 + 26 = 31$, min $5 - 26 = -21$ — $5 \pm 2 \times 13$



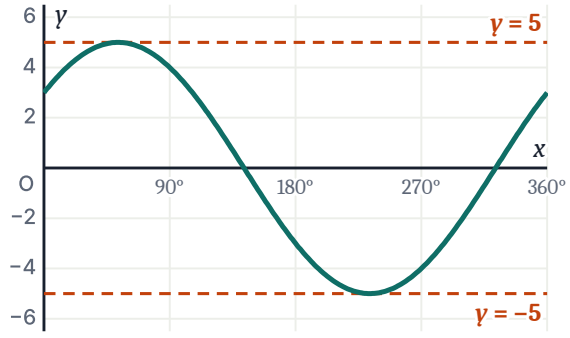

(a): never below 0

## C · Reasoning

**6** For which values of $k$ does $3\cos x + 4\sin x = k$ have solutions?

$5\cos(x - 53.13°) = k$ needs $-1 \le \frac{k}{5} \le 1$. ÷ R; −1 ≤ cos ≤ 1

$-5 \le k \le 5$ — range of k




k between −5 and 5

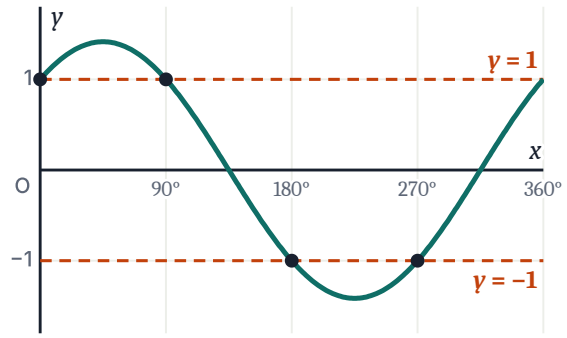
**7** Explain why squaring $\cos x + \sin x = 1$ gives four answers in $0° \le x < 360°$, but only two of them work.

Squaring also solves $\cos x + \sin x = -1$, because $(-1)^2 = 1$.

squaring loses the sign

The extra answers 180° and 270° belong to that equation.

false answers




4 dots: 2 on each line

**8** How many solutions does $5\sin 2x + 12\cos 2x = 6$ have for $0° \le x < 360°$? Explain without solving.

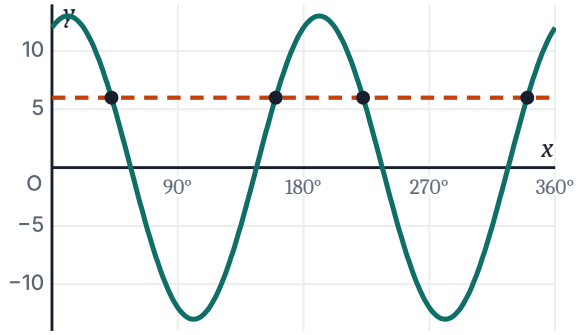
$13\sin(2x + \alpha) = 6$, and $6 < 13$. R-form; $|c| < R$

The angle is $2x$: two complete waves, two crossings each.

period 180°

2 per wave

4 solutions.




4 crossings

## D · Longer problems

**9** Solve $2\sin 2x + \cos 2x = 1$ for $0° \le x < 180°$:
(a) using the double-angle formulae
(b) using the R-form. Check your answers agree.

(a) $4\sin x\cos x + 1 - 2\sin^2 x = 1 \Rightarrow$
$2\sin x(2\cos x - \sin x) = 0$

double angle; factorise

solve each factor

$\sin x = 0$: $x = 0°$; $\tan x = 2$: $x = 63.4°$

(b) $R = \sqrt{4+1} = \sqrt{5}$, $\tan\alpha = \frac{1}{2} \Rightarrow \alpha = 26.57°$:

$\sqrt{5}\sin(2x + 26.57°) = 1$

R-form

$\sin(2x + 26.57°) = \frac{1}{\sqrt{5}}$; $26.57° \le 2x + 26.57° < 386.57°$:

$2x + 26.57° = 26.57°, 153.43°$

÷ R; shift the interval

subtract α; ÷ 2

$2x = 0°, 126.87° \Rightarrow x = 0°, 63.4°$: they agree.

**10** $f(\theta) = 4\cos\theta - 3\sin\theta$.
(a) Write $f(\theta)$ in the form $R\cos(\theta + \alpha)$, where $0° < \alpha < 90°$.
(b) Solve $f(\theta) = 1$ for $0° \le \theta < 360°$.

(a) $R\cos\alpha = 4$, $R\sin\alpha = 3 \Rightarrow R = 5$, $\tan\alpha = \frac{3}{4} \Rightarrow \alpha = 36.87°$

compare: minus gives +α

write the R-form

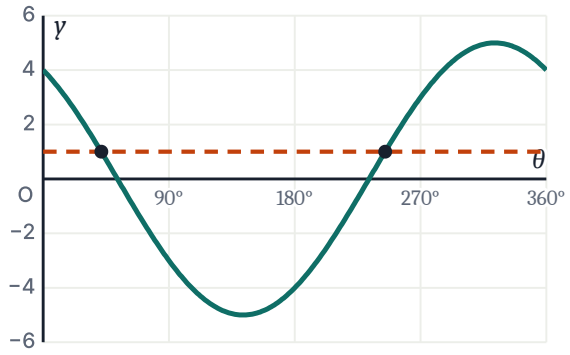
$f(\theta) = 5\cos(\theta + 36.87°)$

(b) $\cos(\theta + 36.87°) = 0.2$; $36.87° \le \theta + 36.87° < 396.87°$:
$\theta + 36.87° = 78.46°, 281.54°$

÷ R; shift the interval

subtract α

$\theta = 41.6°, 244.7°$




check: 2 crossings

**E · Exam-style question** AQA-style. Use the mark scheme to self-assess.

**11** (a) Express $2\cos x - 5\sin x$ in the form $R\cos(x + \alpha)$, where $R > 0$ and $0° < \alpha < 90°$. Give $\alpha$ to 2 decimal places. [3 marks]

(b) Hence solve $2\cos x - 5\sin x = 3$ for $0° \le x < 360°$, giving your answers to 1 decimal place. [4 marks]

(c) Write down the maximum value of $\dfrac{1}{10 + 2\cos x - 5\sin x}$. [2 marks]

**(a)** $R\cos\alpha = 2, R\sin\alpha = 5$ **M1** — compare coefficients

$R = \sqrt{29}$ **A1** — square and add

$\tan\alpha = 2.5 \Rightarrow \alpha = 68.20°$ **A1** — divide: tan α

**(b)** $\cos(x + 68.20°) = \dfrac{3}{\sqrt{29}} = 0.5571$ **M1** — ÷ R

$68.20° \le x + 68.20° < 428.20°$: $x + 68.20° = 303.85°, 416.15°$ **M1** — shift the interval

$x = 235.7°, 347.9°$ **A1 A1** — subtract α

**(c)** Smallest denominator $10 - \sqrt{29}$ **M1** — largest R-form value

Max $= \dfrac{1}{10 - \sqrt{29}} \approx 0.217$ **A1** — small denominator: max