Proving Trigonometric Identities

Specification: AQA 7357: E8, E6, E5 · Edexcel 9MA0: 5.8, 5.6, 5.5

INDEPENDENT

You do

A · One or two steps Prove each identity.

1 (a) $\dfrac{\sin 2x}{\sin x} \equiv 2\cos x$

(b) $\dfrac{\cos 2x + 1}{2\cos x} \equiv \cos x$

(a) LHS $= \dfrac{2\sin x\cos x}{\sin x}$ — sin 2x = 2 sin x cos x

$= 2\cos x$ = RHS — cancel sin x

(b) LHS $= \dfrac{(2\cos^2 x - 1) + 1}{2\cos x}$ — cos 2x = 2 cos²x − 1

$= \dfrac{2\cos^2 x}{2\cos x} = \cos x$ = RHS — cancel 2 cos x

2 (a) $\dfrac{2\sin x\cos x}{\cos^2 x - \sin^2 x} \equiv \tan 2x$

(b) $\dfrac{1 - \cos 2x}{1 + \cos 2x} \equiv \tan^2 x$

(a) $2\sin x\cos x = \sin 2x$ and $\cos^2 x - \sin^2 x = \cos 2x$ — double angle backwards

LHS $= \dfrac{\sin 2x}{\cos 2x} = \tan 2x$ = RHS — sin/cos = tan

(b) $1 - \cos 2x = 1 - (1 - 2\sin^2 x) = 2\sin^2 x$;
$1 + \cos 2x = 1 + (2\cos^2 x - 1) = 2\cos^2 x$ — double angle: top and bottom

LHS $= \dfrac{2\sin^2 x}{2\cos^2 x} = \tan^2 x$ = RHS — sin²/cos² = tan²

3 (a) $\sin 2x \tan x + \cos 2x \equiv 1$

(b) $2\operatorname{cosec} 2x - \cot x \equiv \tan x$

(a) LHS $= 2\sin x\cos x \times \dfrac{\sin x}{\cos x} + (1 - 2\sin^2 x)$ — double angle; tan = sin ÷ cos

$= 2\sin^2 x + 1 - 2\sin^2 x = 1$ = RHS — cancel cos x; collect

(b) LHS $= \dfrac{2}{2\sin x\cos x} - \dfrac{\cos x}{\sin x} = \dfrac{1}{\sin x\cos x} - \dfrac{\cos x}{\sin x}$ — cosec = 1/sin; double angle

$= \dfrac{1 - \cos^2 x}{\sin x\cos x}$ — common denominator

$= \dfrac{\sin^2 x}{\sin x\cos x} = \tan x$ = RHS — sin² = 1 − cos²; cancel

B · Several steps Prove each identity.

4 $\cot x - \cot 2x \equiv \operatorname{cosec} 2x$

LHS $= \dfrac{\cos x}{\sin x} - \dfrac{\cos 2x}{\sin 2x} = \dfrac{2\cos^2 x}{2\sin x\cos x} - \dfrac{\cos 2x}{\sin 2x}$ — common denominator sin 2x

$= \dfrac{2\cos^2 x - (2\cos^2 x - 1)}{\sin 2x} = \dfrac{1}{\sin 2x}$ — double angle: cos form

$= \operatorname{cosec} 2x$ = RHS — 1/sin = cosec

5 $\dfrac{\sec^2 x}{2 - \sec^2 x} \equiv \sec 2x$

Multiply top and bottom by $\cos^2 x$: LHS $= \dfrac{1}{2\cos^2 x - 1}$ — × cos²x top and bottom

$= \dfrac{1}{\cos 2x} = \sec 2x$ = RHS — double angle backwards

6 $\dfrac{\cos x + \sin x}{\cos x - \sin x} \equiv \dfrac{1 + \sin 2x}{\cos 2x}$

Multiply top and bottom by $(\cos x + \sin x)$. — × (cos x + sin x)

LHS $= \dfrac{(\cos x + \sin x)^2}{(\cos x - \sin x)(\cos x + \sin x)}$ — top and bottom

$= \dfrac{\cos^2 x + 2\sin x\cos x + \sin^2 x}{\cos^2 x - \sin^2 x}$ — expand; difference of squares

$= \dfrac{1 + \sin 2x}{\cos 2x}$ = RHS — sin² + cos² = 1; double angle

C · Prove or disprove

7 Prove or disprove: $(\sin x + \cos x)^2 \equiv 1$.

Try $x = 45°$: $(\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2})^2 = (\sqrt{2})^2 = 2 \neq 1$ — test a value

False. In fact $(\sin x + \cos x)^2 \equiv 1 + \sin 2x$. — counter-example

8 Prove or disprove: $\cos 2x \equiv 2\cos x - 1$.

Try $x = 60°$: $\cos 120° = -\frac{1}{2}$, but $2\cos 60° - 1 = 0$. — test a value

False. (It does work at $x = 0°$ and $90°$, so choose your test value with care.) — choose with care

9 Prove or disprove: $\tan 2x \equiv 2\tan x$.

Try $x = 30°$: $\tan 60° = \sqrt{3} \approx 1.73$, but $2\tan 30° = \frac{2}{\sqrt{3}} \approx 1.15$. — test a value

False. — counter-example

D · Addition and triple angles

10 Use $\cos 3x \equiv 4\cos^3 x - 3\cos x$ to prove that $\frac{\cos 3x}{\cos x} \equiv 2\cos 2x - 1$.

LHS $= \frac{4\cos^3 x - 3\cos x}{\cos x} = 4\cos^2 x - 3$ — ÷ cos x

$= 2(2\cos^2 x) - 3 = 2(\cos 2x + 1) - 3$ — cos 2x = 2 cos²x − 1

$= 2\cos 2x - 1$ = RHS — simplify

11 Prove that $\frac{\sin(A+B) + \sin(A-B)}{\cos(A+B) + \cos(A-B)} \equiv \tan A$.

Top

$= (\sin A\cos B + \cos A\sin B) + (\sin A\cos B - \cos A\sin B) = 2\sin A\cos B$ — addition formulae; collect

Bottom

$= (\cos A\cos B - \sin A\sin B) + (\cos A\cos B + \sin A\sin B) = 2\cos A\cos B$ — addition formulae; collect

LHS $= \frac{2\sin A\cos B}{2\cos A\cos B} = \frac{\sin A}{\cos A} = \tan A$ = RHS — cancel 2 cos B; sin/cos = tan

12 Prove that $4\sin(x + 30°)\sin(x - 30°) \equiv 1 - 2\cos 2x$.

LHS

$= 4(\frac{\sqrt{3}}{2}\sin x + \frac{1}{2}\cos x)(\frac{\sqrt{3}}{2}\sin x - \frac{1}{2}\cos x)$ — addition formulae; exact values

$= 4(\frac{3}{4}\sin^2 x - \frac{1}{4}\cos^2 x) = 3\sin^2 x - \cos^2 x$ — difference of squares

$= 3\sin^2 x - (1 - \sin^2 x) = 4\sin^2 x - 1$ — sin² + cos² = 1

$= 2(1 - \cos 2x) - 1 = 1 - 2\cos 2x$ = RHS — double angle: sin form

E · Exam-style question AQA-style. Use the mark scheme to self-assess.

13 (a) Show that $\dfrac{\cos 2\theta}{\cos\theta + \sin\theta} \equiv \cos\theta - \sin\theta$. [2 marks]

(b) Hence solve $\dfrac{\cos 2\theta}{\cos\theta + \sin\theta} = 0.5$ for $0° \le \theta < 360°$. Give your answers to 1 decimal place. [5 marks]

(a) LHS $= \dfrac{\cos^2\theta - \sin^2\theta}{\cos\theta + \sin\theta} = \dfrac{(\cos\theta - \sin\theta)(\cos\theta + \sin\theta)}{\cos\theta + \sin\theta}$ **M1** — difference of two squares

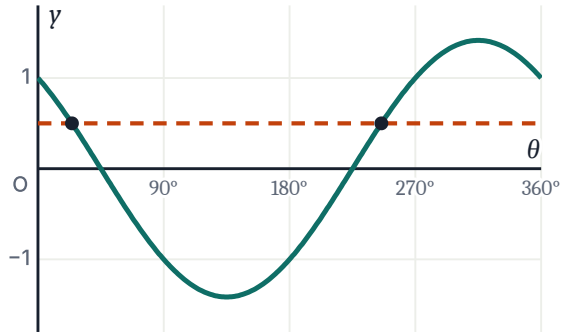
$= \cos\theta - \sin\theta$ **A1** — cancel

(b) $\cos\theta - \sin\theta = 0.5$: $\sqrt{2}\cos(\theta + 45°) = 0.5$ **M1** — R-form

$\cos(\theta + 45°) = 0.3536$; $45° \le \theta + 45° < 405°$ **M1** — ÷ R; shift the interval

$\theta + 45° = 69.30°, 290.70°$ **A1** — calculator; 360° − x

$\theta = 24.3°, 245.7°$ **A1 A1** — neither makes cos θ + sin θ = 0



check: 2 crossings